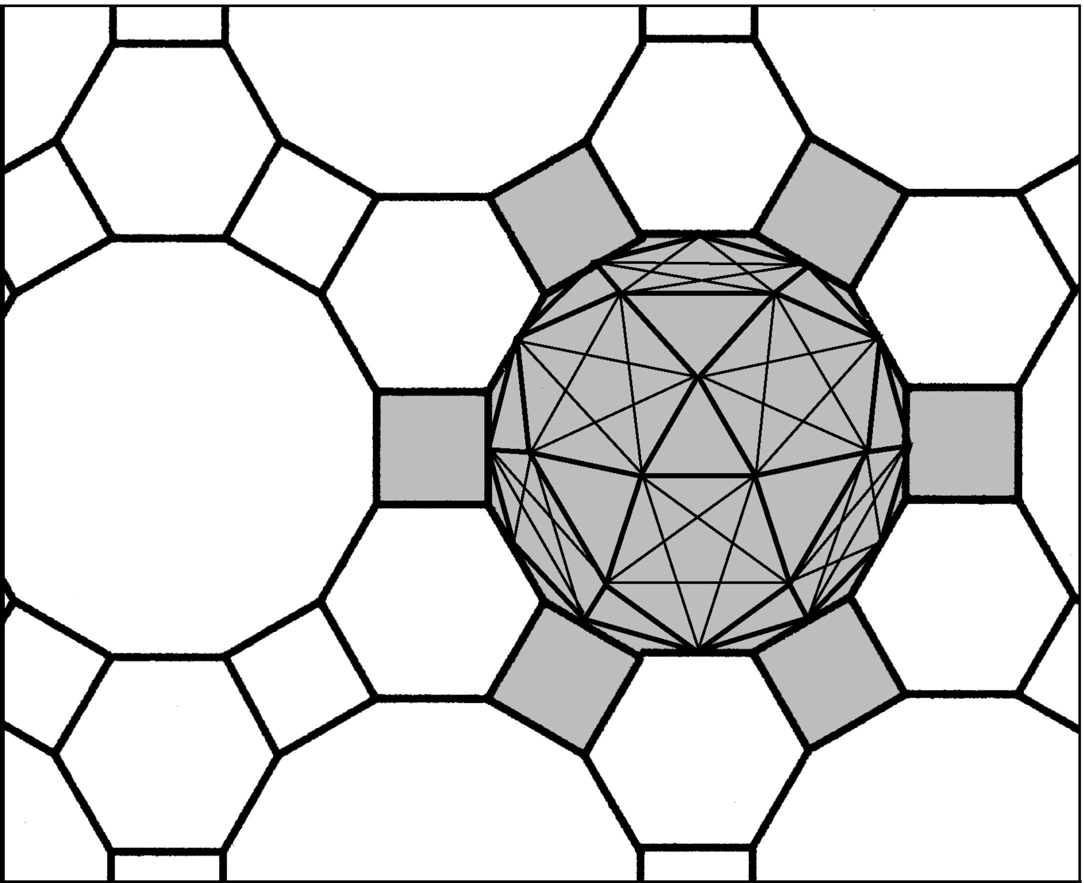
Workbook 1



Basic Algebra

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

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HELM Workbooks List

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Contents 1

Basic Algebra

Learning outcomes

In this Workbook you will learn about some of the basic building blocks of mathematics. As well as becoming familiar with the notation and symbols used in mathematics you will learn the fundamental rules of algebra upon which much of mathematics is based. In particular you will learn about indices and how to simplify algebraic expressions, using a variety of approaches: collecting like terms, removing brackets and factorisation. Finally, you will learn how to transpose formulae.

Mathematical Notation and Symbols

1.1

Introduction

This introductory Section reminds you of important notations and conventions used throughout engineering mathematics. We discuss the arithmetic of numbers, the plus or minus sign, $\pm$, the modulus notation $|\ |$, and the factorial notation !. We examine the order in which arithmetical operations are carried out. Symbols are introduced to represent physical quantities in formulae and equations. The topic of algebra deals with the manipulation of these symbols. The Section closes with an introduction to algebraic conventions. In what follows a working knowledge of the addition, subtraction, multiplication and division of numerical fractions is essential.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide fractions
- be able to express fractions in equivalent forms

Learning Outcomes

On completion you should be able to ...

- recognise and use a wide range of common mathematical symbols and notations

1. Numbers, operations and common notations

A knowledge of the properties of numbers is fundamental to the study of engineering mathematics. Students who possess this knowledge will be well-prepared for the study of algebra. Much of the terminology used throughout the rest of this Section can be most easily illustrated by applying it to numbers. For this reason we strongly recommend that you work through this Section even if the material is familiar.

The number line

A useful way of picturing numbers is to use a **number line**. Figure 1 shows part of this line. Positive numbers are represented on the right-hand side of this line, negative numbers on the left-hand side. Any whole or fractional number can be represented by a point on this line which is also called the **real number line**, or simply the **real line**. Study Figure 1 and note that a minus sign is always used to indicate that a number is negative, whereas the use of a plus sign is optional when describing positive numbers.

The line extends indefinitely both to the left and to the right. Mathematically we say that the line extends from minus infinity to plus infinity. The symbol for **infinity** is ∞.

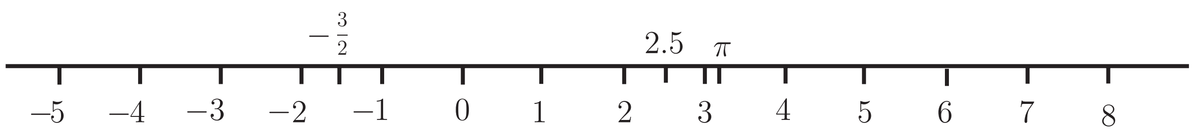


Figure 1: Numbers can be represented on a number line

The symbol $>$ means 'greater than'; for example $6 > 4$. Given any number, all numbers to the right of it on the number line are greater than the given number. The symbol $<$ means 'less than'; for example $-3 < 19$. We also use the symbols $\geq$ meaning 'greater than or equal to' and $\leq$ meaning 'less than or equal to'. For example, $7 \leq 10$ and $7 \leq 7$ are both true statements.

Sometimes we are interested in only a small section, or **interval**, of the real line. We write $[1, 3]$ to denote all the real numbers between 1 and 3 inclusive, that is 1 and 3 are included in the interval. Therefore the interval $[1, 3]$ consists of all real numbers x, such that $1 \leq x \leq 3$. The square brackets, $[,]$ mean that the end-points are included in the interval and such an interval is said to be **closed**. We write $(1, 3)$ to represent all real numbers between 1 and 3, but not including the end-points. Thus $(1, 3)$ means all real numbers x such that $1 < x < 3$, and such an interval is said to be **open**. An interval may be closed at one end and open at the other. For example, $(1, 3]$ consists of all numbers x such that $1 < x \leq 3$. Intervals can be represented on a number line. A **closed end-point** is denoted by $\bullet$; an **open end-point** is denoted by $\circ$. The intervals $(-6, -4)$, $[-1, 2]$ and $(3, 4]$ are illustrated in Figure 2.

Figure 2: The intervals $(-6, -4)$, $[-1, 2]$ and $(3, 4]$ depicted on the real line

2. Calculation with numbers

To perform calculations with numbers we use the **operations**, $+$, $-$, $\times$ and $\div$.

Addition ($+$)

We say that $4+5$ is the **sum** of 4 and 5. Note that $4+5$ is equal to $5+4$ so that the order in which we write down the numbers does not matter when we are adding them. Because the order does not matter, addition is said to be **commutative**. This first property is called **commutativity**.

When more than two numbers are to be added, as in $4+8+9$, it makes no difference whether we add the 4 and 8 first to get $12+9$, or whether we add the 8 and 9 first to get $4+17$. Whichever way we work we will obtain the same result, 21. Addition is said to be **associative**. This second property is called **associativity**.

Subtraction ($-$)

We say that $8-3$ is the **difference** of 8 and 3. Note that $8-3$ is *not* the same as $3-8$ and so the order in which we write down the numbers is important when we are subtracting them i.e. subtraction is not commutative. Subtracting a negative number is equivalent to adding a positive number, thus $7-(-3)=7+3=10$.

The plus or minus sign ($\pm$)

In engineering calculations we often use the notation **plus or minus**, $\pm$. For example, we write 12 ± 8 as shorthand for the two numbers $12+8$ and $12-8$, that is 20 and 4. If we say a number lies in the range 12 ± 8 we mean that the number can lie between 4 and 20 inclusive.

Multiplication ($\times$)

The instruction to multiply, or obtain the product of, the numbers 6 and 7 is written 6×7. Sometimes the multiplication sign is missed out altogether and we write $(6)(7)$.

Note that $(6)(7)$ is the same as $(7)(6)$ so multiplication of numbers is commutative. If we are multiplying three numbers, as in $2\times 3\times 4$, we obtain the same result whether we multiply the 2 and 3 first to obtain 6×4, or whether we multiply the 3 and 4 first to obtain 2×12. Either way the result is 24. Multiplication of numbers is associative.

Recall that when multiplying positive and negative numbers the sign of the result is given by the rules given in Key Point 1.

Key Point 1

Multiplication

When multiplying numbers:

positive $\times$ positive $=$ positive	negative $\times$ negative $=$ positive
positive $\times$ negative $=$ negative	negative $\times$ positive $=$ negative

For example, $(-4) \times 5 = -20$, and $(-3) \times (-6) = 18$.

When dealing with fractions we sometimes use the word 'of' as in 'find $\frac{1}{2}$ of 36'. In this context 'of' is equivalent to multiply, that is

$$\frac{1}{2} \text{ of } 36 \quad \text{is equivalent to} \quad \frac{1}{2} \times 36 = 18$$

Division (÷) or (/)

The quantity $8 \div 4$ means 8 divided by 4. This is also written as $8/4$ or $\frac{8}{4}$ and is known as the **quotient** of 8 and 4. In the fraction $\frac{8}{4}$ the top line is called the **numerator** and the bottom line is called the **denominator**. Note that $8/4$ is not the same as $4/8$ and so the order in which we write down the numbers is important. Division is not commutative.

When dividing positive and negative numbers, recall the following rules in Key Point 2 for determining the sign of the result:

Key Point 2

Division

When dividing numbers:

$$\frac{\text{positive}}{\text{positive}} = \text{positive} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

$$\frac{\text{negative}}{\text{positive}} = \text{negative} \qquad \frac{\text{negative}}{\text{negative}} = \text{positive}$$

The reciprocal of a number

The **reciprocal** of a number is found by inverting it. If the number $\frac{2}{3}$ is **inverted** we get $\frac{3}{2}$. So the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Because we can write 4 as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

Task State the reciprocal of (a) $\frac{6}{11}$, (b) $\frac{1}{5}$, (c) -7.

Your solution

(a) (b) (c)

Answer

(a) $\frac{11}{6}$ (b) $\frac{5}{1}$ (c) $-\frac{1}{7}$

The modulus notation (| |)

We shall make frequent use of the modulus notation $|\ |$. The **modulus** of a number is the size of that number regardless of its sign. For example $|4|$ is equal to 4, and $|-3|$ is equal to 3. The modulus of a number is thus never negative.

Task State the modulus of (a) -17, (b) $\frac{1}{5}$, (c) $-\frac{1}{7}$ (d) 0.

Your solution

(a) (b) (c) (d)

Answer

The modulus of a number is found by ignoring its sign. (a) 17 (b) $\frac{1}{5}$ (c) $\frac{1}{7}$ (d) 0

The factorial symbol (!)

Another commonly used notation is the **factorial**, denoted by the exclamation mark '!'. The number $5!$, read 'five factorial', or 'factorial five', is a shorthand notation for the expression $5\times4\times3\times2\times1$, and the number $7!$ is shorthand for $7\times6\times5\times4\times3\times2\times1$. Note that $1!$ equals 1, and by convention $0!$ is defined as 1 also. Your scientific calculator is probably able to evaluate factorials of small integers. It is important to note that factorials only apply to positive integers.

Key Point 3

Factorial notation

If n is a positive integer then $n! = n\times(n-1)\times(n-2)\ldots5\times4\times3\times2\times1$

Example 1

(a) Evaluate 4! and 5! without using a calculator.

(b) Use your calculator to find $10!$.

Solution

(a) 4! $= 4 \times 3 \times 2 \times 1 = 24$. Similarly, 5! $= 5 \times 4 \times 3 \times 2 \times 1 = 120$. Note that $5! = 5 \times 4! = 5 \times 24 = 120$.

(b) $10! = 3,628,800$.

Task

Find the factorial button on your calculator and hence compute 11!.
(The button may be marked ! or n!). Check that $11! = 11 \times 10!$

Your solution

$11! =$ $\qquad\qquad$ $11 \times 10! =$

Answer

$11! = 39916800$

$11 \times 10! = 11 \times 3628800 = 39916800$

3. Rounding to n decimal places

In general, a calculator or computer is unable to store every decimal place of a real number. Real numbers are **rounded**. To round a number to n decimal places we look at the $(n+1)^{th}$ digit in the decimal expansion of the number.

- If the $(n+1)^{th}$ digit is 0, 1, 2, 3 or 4 then we **round down**: that is, we simply chop to n places. (In other words we neglect the $(n+1)^{th}$ digit and any digits to its right.)
- If the $(n+1)^{th}$ digit is 5, 6, 7, 8 or 9 then we **round up**: we add 1 to the n^{th} decimal place and then chop to n places.

For example

$$\frac{1}{3} = 0.3333 \qquad \text{rounded to 4 decimal places}$$

$$\frac{8}{3} = 2.66667 \qquad \text{rounded to 5 decimal places}$$

$$\pi = 3.142 \qquad \text{rounded to 3 decimal places}$$

$$2.3403 = 2.340 \qquad \text{rounded to 3 decimal places}$$

Sometimes the phrase 'decimal places' is abbreviated to 'd.p.' or 'dec.pl.'.

Example 2

Write down each of these numbers rounded to 4 decimal places:
0.12345, -0.44444, 0.5555555, 0.000127351, 0.000005, 123.456789

Solution

0.1235, -0.4444, 0.5556, 0.0001, 0.0000, 123.4568

Task

Write down each of these numbers, rounded to 3 decimal places:
0.87264, 0.1543, 0.889412, -0.5555, 45.6789, 6.0003

Your solution

Answer

0.873, 0.154, 0.889, -0.556, 45.679, 6.000

4. Rounding to n significant figures

This process is similar to rounding to decimal places but there are some subtle differences.

To round a number to n significant figures we look at the $(n+1)^{th}$ digit in the decimal expansion of the number.

- If the $(n+1)^{th}$ digit is 0, 1, 2, 3 or 4 then we **round down**: that is, we simply chop to n places, inserting zeros if necessary before the decimal point. (In other words we neglect the $(n+1)^{th}$ digit and any digits to its right.)
- If the $(n+1)^{th}$ digit is 5, 6, 7, 8 or 9 then we **round up**: we add 1 to the n^{th} decimal place and then chop to n places, inserting zeros if necessary before the decimal point.

Examples are given on the next page.

$$\frac{1}{3} = 0.3333 \quad \text{rounded to 4 significant figures}$$

$$\frac{8}{3} = 2.66667 \quad \text{rounded to 6 significant figures}$$

$$\pi = 3.142 \quad \text{rounded to 4 significant figures}$$

$$2136 = 2000 \quad \text{rounded to 1 significant figure}$$

$$36.78 = 37 \quad \text{rounded to 2 significant figures}$$

$$6.2399 = 6.240 \quad \text{rounded to 4 significant figures}$$

Sometimes the phrase "significant figures" is abbreviated as "s.f." or "sig.fig."

Example 3

Write down each of these numbers, rounding them to 4 significant figures:
0.12345, -0.44444, 0.5555555, 0.000127351, 25679, 123.456789, 3456543

Solution

0.1235, -0.4444, 0.5556, 0.0001274, 25680, 123.5, 3457000

Task

Write down each of these numbers rounded to 3 significant figures:
0.87264, 0.1543, 0.889412, -0.5555, 2.346, 12343.21, 4245321

Your solution

Answer

0.873, 0.154, 0.889, -0.556, 2.35, 12300, 4250000

Arithmetical expressions

A quantity made up of numbers and one or more of the operations $+$, $-$, $\times$ and $/$ is called an **arithmetical expression**. Frequent use is also made of brackets, or **parentheses**, ( ), to separate different parts of an expression. When evaluating an expression it is conventional to evaluate quantities within brackets first. Often a division line implies bracketed quantities. For example in the expression $\dfrac{3+4}{7+9}$ there is implied bracketing of the numerator and denominator i.e. the expression is $\dfrac{(3+4)}{(7+9)}$ and the bracketed quantities would be evaluated first resulting in the number $\dfrac{7}{16}$.

The BODMAS rule

When several arithmetical operations are combined in one expression we need to know in which order to perform the calculation. This order is found by applying rules known as **precedence rules** which specify which operation has priority. The convention is that bracketed expressions are evaluated first. Any multiplications and divisions are then performed, and finally any additions and subtractions. For short, this is called the BODMAS rule.

Key Point 4

The BODMAS rule

Brackets, ( )	First priority: evaluate terms within brackets
Of, $\times$ **D**ivision, $\div$ **M**ultiplication, $\times$	Second priority: carry out all multiplications and divisions
Addition, $+$ **S**ubtraction, $-$	Third priority: carry out all additions and subtractions

If an expression contains only multiplication and division we evaluate by working from left to right. Similarly, if an expression contains only addition and subtraction we evaluate by working from left to right. In Section 1.2 we will meet another operation called exponentiation, or raising to a power. We shall see that, in the simplest case, this operation is repeated multiplication and it is usually carried out once any brackets have been evaluated.

Example 4

Evaluate $4 - 3 + 7 \times 2$

Solution

The BODMAS rule tells us to perform the multiplication before the addition and subtraction. Thus

$$4 - 3 + 7 \times 2 = 4 - 3 + 14$$

Finally, because the resulting expression contains just addition and subtraction we work from the left to the right, that is

$$4 - 3 + 14 = 1 + 14 = 15$$

Task

Evaluate $4 + 3 \times 7$ using the BODMAS rule to decide which operation to carry out first.

Your solution

$4 + 3 \times 7 =$

Answer

25 (Multiplication has a higher priority than addition.)

Task

Evaluate $(4 - 2) \times 5$.

Your solution

$(4 - 2) \times 5 =$

Answer

$2 \times 5 = 10$. (The bracketed quantity must be evaluated first.)

Example 5

Evaluate $8 \div 2 - (4 - 5)$

Solution

The bracketed expression is evaluated first:

$$8 \div 2 - (4 - 5) = 8 \div 2 - (-1)$$

Division has higher priority than subtraction and so this is carried out next giving

$$8 \div 2 - (-1) = 4 - (-1)$$

Subtracting a negative number is equivalent to adding a positive number. Thus

$$4 - (-1) = 4 + 1 = 5$$

Evaluate $\dfrac{9-4}{25-5}$.

(Remember that the dividing line implies that brackets are present around the numerator and around the denominator.)

Your solution

Answer

$$\frac{9-4}{25-5} = \frac{(9-4)}{(25-5)} = \frac{5}{20} = \frac{1}{4}$$

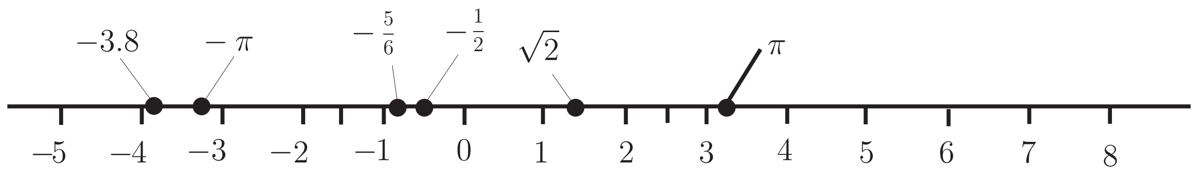
Exercises

1. Draw a number line and on it label points to represent -5, -3.8, $-\pi$, $-\dfrac{5}{6}$, $-\dfrac{1}{2}$, 0, $\sqrt{2}$, π, 5.
2. Simplify without using a calculator (a) -5×-3, (b) -5×3, (c) 5×-3, (d) 15×-4, (e) -14×-3, (f) $\dfrac{18}{-3}$, (g) $\dfrac{-21}{7}$, (h) $\dfrac{-36}{-12}$.
3. Evaluate (a) $3+2\times 6$, (b) $3-2-6$, (c) $3+2-6$, (d) $15-3\times 2$, (e) $15\times 3-2$, (f) $(15\div 3)+2$, (g) $15\div 3+2$, (h) $7+4-11-2$, (i) $7\times 4+11\times 2$, (j) $-(-9)$, (k) $7-(-9)$, (l) $-19-(-7)$, (m) $-19+(-7)$.
4. Evaluate (a) $|-18|$, (b) $|4|$, (c) $|-0.001|$, (d) $|0.25|$, (e) $|0.01-0.001|$, (f) $2!$, (g) $8!-3!$, (h) $\dfrac{9!}{8!}$.
5. Evaluate (a) $8+(-9)$, (b) $18-(-8)$, (c) $-18+(-2)$, (d) $-11-(-3)$
6. State the reciprocal of (a) 8, (b) $\dfrac{9}{13}$.
7. Evaluate (a) 7 ± 3, (b) 16 ± 7, (c) $-15\pm\dfrac{1}{2}$, (d) -16 ± 0.05, (e) $|-8|\pm 13$, (f) $|-2|\pm 8$.
8. Which of the following statements are true ?

   (a) $-8\le 8$, (b) $-8\le -8$, (c) $-8\le |8|$, (d) $|-8|<8$, (e) $|-8|\le -8$, (f) $9!\le 8!$, (g) $8!\le 10!$.
9. Explain what is meant by saying that addition of numbers is (a) associative, (b) commutative. Give examples.
10. Explain what is meant by saying that multiplication of numbers is (a) associative, (b) commutative. Give examples.

Answers

1.



2. (a) 15, (b) -15, (c) -15, (d) -60, (e) 42, (f) -6, (g) -3, (h) 3.

3. (a) 15, (b) -5, (c) -1, (d) 9, (e) 43, (f) 7, (g) 7, (h) -2, (i) 50, (j) 9, (k) 16, (l) -12, (m) -26

4. (a) 18, (b) 4, (c) 0.001, (d) 0.25, (e) 0.009, (f) 2, (g) 40314, (h) 9,

5. (a) -1, (b) 26, (c) -20, (d) -8

6. (a) $\dfrac{1}{8}$, (b) $\dfrac{13}{9}$.

7. (a) 4,10, (b) 9,23, (c) $-15\dfrac{1}{2}$, $-14\dfrac{1}{2}$, (d) $-16.05, -15.95$, (e) $-5, 21$, (f) $-6, 10$

8. (a), (b), (c), (g) are true.

9. For example (a) $(1+2)+3 = 1+(2+3)$, and both are equal to 6. (b) $8+2 = 2+8$.

10. For example (a) $(2\times 6)\times 8 = 2\times(6\times 8)$, and both are equal to 96. (b) $7\times 5 = 5\times 7$.

5. Using symbols

Mathematics provides a very rich language for the communication of engineering concepts and ideas, and a set of powerful tools for the solution of engineering problems. In order to use this language it is essential to appreciate how **symbols** are used to represent physical quantities, and to understand the rules and conventions which have been developed to manipulate these symbols.

The choice of which letters or other symbols to use is largely up to the user although it is helpful to choose letters which have some meaning in any particular context. For instance if we wish to choose a symbol to represent the temperature in a room we might use the capital letter T. Similarly the lower case letter t is often used to represent time. Because both time and temperature can vary we refer to T and t as **variables**.

In a particular calculation some symbols represent fixed and unchanging quantities and we call these **constants**. Often we reserve the letters x, y and z to stand for variables and use the earlier letters of the alphabet, such as a, b and c, to represent constants. The Greek letter pi, written π, is used to represent the constant 3.14159.... which appears for example in the formula for the area of a circle. Other Greek letters are frequently used as symbols, and for reference, the Greek alphabet is given in Table 1.

Table 1: The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	Λ	λ	lambda	T	τ	tau
Γ	γ	gamma	K	κ	kappa	Σ	σ	sigma
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Mathematics is a very precise language and care must be taken to note the exact position of any symbol in relation to any other. If x and y are two symbols, then the quantities xy, x^y, x_y can all mean different things. In the expression x^y you will note that the symbol y is placed to the right of and slightly higher than the symbol x. In this context y is called a **superscript**. In the expression x_y, y is placed lower than and to the right of x, and is called a **subscript**.

Example The temperature in a room is measured at four points as shown in Figure 3.

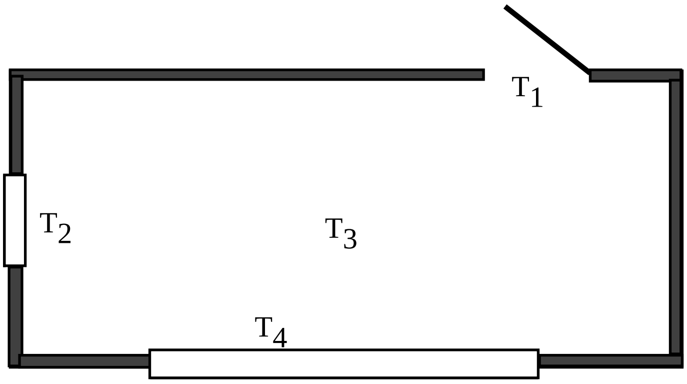


Figure 3: The temperature is measured at four points

Rather than use different letters to represent the four measurements we can use one symbol, T, together with four subscripts to represent the temperature. Thus the four measurements are denoted by T_1, T_2, T_3 and T_4.

6. Combining numbers together using $+, -, \times, \div$

Addition ($+$)

If the letters x and y represent two numbers, then their **sum** is written as $x + y$. Note that $x + y$ is the same as $y + x$ just as $4 + 7$ is equal to $7 + 4$.

Subtraction ($-$)

Subtracting y from x yields $x - y$. Note that $x - y$ is not the same as $y - x$ just as $11 - 7$ is not the same as $7 - 11$, however in both cases the difference is said to be 4.

Multiplication ($\times$)

The instruction to multiply x and y together is written as $x \times y$. Usually the multiplication sign is omitted and we write simply xy. An alternative notation is to use a dot to represent multiplication and so we could write $x.y$ The quantity xy is called the **product** of x and y. As discussed earlier multiplication is both commutative and associative:

i.e. $\quad x \times y = y \times x \quad$ and $\quad (x \times y) \times z = x \times (y \times z)$

This last expression can thus be written $x \times y \times z$ without ambiguity. When mixing numbers and symbols it is usual to write the numbers first. Thus $3 \times x \times y \times 4 = 3 \times 4 \times x \times y = 12xy$.

Example 6

Simplify (a) $9(2y)$, (b) $-3(5z)$, (c) $4(2a)$, (d) $2x \times (2y)$.

Solution

(a) Note that $9(2y)$ means $9 \times (2 \times y)$. Because of the associativity of multiplication $9 \times (2 \times y)$ means the same as $(9 \times 2) \times y$, that is $18y$.

(b) $-3(5z)$ means $-3 \times (5 \times z)$. Because of associativity this is the same as $(-3 \times 5) \times z$, that is $-15z$.

(c) $4(2a)$ means $4 \times (2 \times a)$. We can write this as $(4 \times 2) \times a$, that is $8a$.

(d) Because of the associativity of multiplication, the brackets are not needed and we can write $2x \times (2y) = 2x \times 2y$ which equals

$$2 \times x \times 2 \times y = 2 \times 2 \times x \times y = 4xy.$$

Example 7

What is the distinction between $9(-2y)$ and $9 - 2y$?

Solution

The expression $9(-2y)$ means $9 \times (-2y)$. Because of associativity of multiplication we can write this as $9 \times (-2) \times y$ which equals $-18y$.

On the other hand $9 - 2y$ means subtract $2y$ from 9. This cannot be simplified.

Division ($\div$)

The quantity $x \div y$ means x divided by y. This is also written as x/y or $\dfrac{x}{y}$ and is known as the **quotient** of x and y. In the expression $\dfrac{x}{y}$ the symbol x is called the **numerator** and the symbol y is called the **denominator**. Note that x/y is not the same as y/x. Division by 1 leaves a quantity unchanged so that $\dfrac{x}{1}$ is simply x.

Algebraic expressions

A quantity made up of symbols and the operations $+$, $-$, $\times$ and $/$ is called an **algebraic expression**. One algebraic expression divided by another is called an algebraic fraction. Thus

$$\frac{x+7}{x-3} \qquad \text{and} \qquad \frac{3x-y}{2x+z}$$

are algebraic fractions. The **reciprocal** of an algebraic fraction is found by inverting it. Thus the reciprocal of $\dfrac{2}{x}$ is $\dfrac{x}{2}$. The reciprocal of $\dfrac{x+7}{x-3}$ is $\dfrac{x-3}{x+7}$.

Example 8

State the reciprocal of each of the following expressions:

(a) $\dfrac{y}{z}$, (b) $\dfrac{x+z}{a-b}$, (c) $3y$, (d) $\dfrac{1}{a+2b}$, (e) $-\dfrac{1}{y}$

Solution

(a) $\dfrac{z}{y}$.

(b) $\dfrac{a-b}{x+z}$.

(c) $3y$ is the same as $\dfrac{3y}{1}$ so the reciprocal of $3y$ is $\dfrac{1}{3y}$.

(d) The reciprocal of $\dfrac{1}{a+2b}$ is $\dfrac{a+2b}{1}$ or simply $a+2b$.

(e) The reciprocal of $-\dfrac{1}{y}$ is $-\dfrac{y}{1}$ or simply $-y$.

Finding the reciprocal of complicated expressions can cause confusion. Study the following Example carefully.

Example 9

Obtain the reciprocal of:

(a) $p+q$, (b) $\dfrac{1}{R_1}+\dfrac{1}{R_2}$

Solution

(a) Because $p+q$ can be thought of as $\dfrac{p+q}{1}$ its reciprocal is $\dfrac{1}{p+q}$. Note in particular that the reciprocal of $p+q$ is **not** $\dfrac{1}{p}+\dfrac{1}{q}$. This distinction is important and a common cause of error. To avoid an error carefully identify the numerator and denominator in the original expression before inverting.

(b) The reciprocal of $\dfrac{1}{R_1}+\dfrac{1}{R_2}$ is $\dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$. To simplify this further requires knowledge of the addition of algebraic fractions which is dealt with in HELM 1.4. It is important to note that the reciprocal of $\dfrac{1}{R_1}+\dfrac{1}{R_2}$ is **not** R_1+R_2.

The equals sign (=)

The equals sign, $=$, is used in several different ways.

Firstly, an equals sign is used in **equations**. The left-hand side and right-hand side of an equation are equal only when the variable involved takes specific values known as **solutions** of the equation. For example, in the equation $x-8=0$, the variable is x. The left-hand side and right-hand side are only equal when x has the value 8. If x has any other value the two sides are not equal.

Secondly, the equals sign is used in **formulae**. Physical quantities are often related through a formula. For example, the formula for the length, C, of the circumference of a circle expresses the relationship between the circumference of the circle and its radius, r. This formula states $C=2\pi r$. When used in this way the equals sign expresses the fact that the quantity on the left is found by evaluating the expression on the right.

Thirdly, an equals sign is used in **identities**. An identity looks just like an equation, but it is true for *all* values of the variable. We shall see shortly that $(x-1)(x+1)=x^2-1$ for any value of x whatsoever. This mean that the quantity on the left means exactly the same as that on the right whatever the value of x. To distinguish this usage from other uses of the equals symbol it is more correct to write $(x-1)(x+1)\equiv x^2-1$, where $\equiv$ means 'is identically equal to'. However, in practice, the equals sign is often used. We will only use $\equiv$ where it is particularly important to do so.

The 'not equals' sign ($\neq$)

The sign $\neq$ means 'is not equal to'. For example, $5 \neq 6$, $7 \neq -7$.

The notation for the change in a variable (δ)

The **change** in the value of a quantity is found by subtracting its initial value from its final value. For example, if the temperature of a mixture is initially $13°$C and at a later time is found to be $17°$C, the change in temperature is $17 - 13 = 4°$C. The Greek letter δ is often used to indicate such a change. If x is a variable we write δx to stand for a change in the value of x. We sometimes refer to δx as an **increment** in x. For example if the value of x changes from 3 to 3.01 we could write $\delta x = 3.01 - 3 = 0.01$. It is important to note that this is **not** the product of δ and x, rather the whole symbol 'δx' means 'the increment in x'.

Sigma (or summation) notation ($\sum$)

This provides a concise and convenient way of writing long sums.
The sum

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_{11} + x_{12}$$

is written using the capital Greek letter sigma, $\sum$, as

$$\sum_{k=1}^{12} x_k$$

The symbol $\sum$ stands for the sum of all the values of x_k as k ranges from 1 to 12. Note that the lower-most and upper-most values of k are written at the bottom and top of the sigma sign respectively.

Example 10

Write out explicitly what is meant by $\displaystyle\sum_{k=1}^{5} k^3$.

Solution

We must let k range from 1 to 5. $\displaystyle\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

Task

Express $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ concisely using sigma notation.

Each term has the form $\frac{1}{k}$ where k varies from 1 to 4. Write down the sum using the sigma notation:

Your solution

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$

Answer

$\displaystyle\sum_{k=1}^{4}\frac{1}{k}$

Example 11

Write out explicitly (a) $\displaystyle\sum_{k=1}^{3} 1$, (b) $\displaystyle\sum_{k=0}^{4} 2$.

Solution

(a) Here k does not appear explicitly in the terms to be added. This means add the constant 1, three times.

$$\sum_{k=1}^{3} 1 = 1+1+1 = 3$$

In general $\displaystyle\sum_{k=1}^{n} 1 = n$.

(b) Here k starts at zero so there are $n+1$ terms where $n=4$:

$$\sum_{k=0}^{4} 2 = 2+2+2+2+2 = 10$$

Exercises

1. State the reciprocal of (a) x, (b) $\dfrac{1}{z}$, (c) xy, (d) $\dfrac{1}{xy}$, (e) $a+b$, (f) $\dfrac{2}{a+b}$

2. The pressure p in a reaction vessel changes from 35 pascals to 38 pascals. Write down the value of δp.

3. Express as simply as possible (a) $(-3) \times x \times (-2) \times y$, (b) $9 \times x \times z \times (-5)$.

4. Simplify (a) $8(2y)$, (b) $17x(-2y)$, (c) $5x(8y)$, (d) $5x(-8y)$

5. What is the distinction between $5x(2y)$ and $5x - 2y$?

6. The value of x is 100 ± 3. The value of y is 120 ± 5. Find the maximum and minimum values of

   (a) $x+y$, (b) xy, (c) $\dfrac{x}{y}$, (d) $\dfrac{y}{x}$.

7. Write out explicitly (a) $\displaystyle\sum_{i=1}^{n} f_i$, (b) $\displaystyle\sum_{i=1}^{n} f_i x_i$.

8. By writing out the terms explicitly show that $\displaystyle\sum_{k=1}^{5} 3k = 3\sum_{k=1}^{5} k$

9. Write out explicitly $\displaystyle\sum_{k=1}^{3} y(x_k)\delta x_k$.

Answers

1. (a) $\dfrac{1}{x}$, (b) z, (c) $\dfrac{1}{xy}$, (d) xy, (e) $\dfrac{1}{a+b}$, (f) $\dfrac{a+b}{2}$.

2. $\delta p = 3$ pascals.

3. (a) $6xy$, (b) $-45xz$

4. (a) $16y$, (b) $-34xy$, (c) $40xy$, (d) $-40xy$

5. $5x(2y) = 10xy$, $5x - 2y$ cannot be simplified.

6. (a) max 228, min 212, (b) 12875, 11155, (c) 0.8957, 0.7760, (d) 1.2887, 1.1165

7. (a) $\displaystyle\sum_{i=1}^{n} f_i = f_1 + f_2 + \ldots + f_{n-1} + f_n$,

   (b) $\displaystyle\sum_{i=1}^{n} f_i x_i = f_1 x_1 + f_2 x_2 + \ldots + f_{n-1}x_{n-1} + f_n x_n$.

9. $y(x_1)\delta x_1 + y(x_2)\delta x_2 + y(x_3)\delta x_3$.

Indices

1.2

Introduction

Indices, or powers, provide a convenient notation when we need to multiply a number by itself several times. In this Section we explain how indices are written, and state the rules which are used for manipulating them.

Expressions built up using non-negative whole number powers of a variable – known as polynomials – occur frequently in engineering mathematics. We introduce some common polynomials in this Section.

Finally, scientific notation is used to express very large or very small numbers concisely. This requires use of indices. We explain how to use scientific notation towards the end of the Section.

Prerequisites

Before starting this Section you should ...

- be familiar with algebraic notation and symbols

Learning Outcomes

On completion you should be able to ...

- perform calculations using indices
- state and use the laws of indices
- use scientific notation

1. Index notation

The number $4 \times 4 \times 4$ is written, for short, as 4^3 and read '4 raised to the power 3' or '4 cubed'. Note that the number of times '4' occurs in the product is written as a superscript. In this context we call the superscript 3 an **index** or **power**. Similarly we could write

$$5 \times 5 = 5^2, \text{ read '5 to the power 2' or '5 squared'}$$

and

$$7 \times 7 \times 7 \times 7 \times 7 = 7^5 \qquad a \times a \times a = a^3, \qquad m \times m \times m \times m = m^4$$

More generally, in the expression x^y, x is called the **base** and y is called the index or power. The plural of index is **indices**. The process of raising to a power is also known as **exponentiation** because yet another name for a power is an **exponent**. When dealing with numbers your calculator is able to evaluate expressions involving powers, probably using the x^y button.

Example 12

Use a calculator to evaluate 3^{12}.

Solution

Using the x^y button on the calculator check that you obtain $3^{12} = 531441$.

Example 13

Identify the index and base in the following expressions. (a) 8^{11}, (b) $(-2)^5$, (c) p^{-q}

Solution

(a) In the expression 8^{11}, 8 is the base and 11 is the index.

(b) In the expression $(-2)^5$, -2 is the base and 5 is the index.

(c) In the expression p^{-q}, p is the base and $-q$ is the index. The interpretation of a negative index will be given in sub-section 4 which starts on page 31.

Recall from Section 1.1 that when several operations are involved we can make use of the BODMAS rule for deciding the order in which operations must be carried out. The BODMAS rule makes no mention of exponentiation. Exponentiation should be carried out immediately after any brackets have been dealt with and before multiplication and division. Consider the following examples.

Example 14

Evaluate 7×3^2.

Solution

There are two operations involved here, exponentiation and multiplication. The exponentiation should be carried out before the multiplication. So $7 \times 3^2 = 7 \times 9 = 63$.

Example 15

Write out fully (a) $3m^4$, (b) $(3m)^4$.

Solution

(a) In the expression $3m^4$ the exponentiation is carried out before the multiplication by 3. So

$$3m^4 \quad \text{means} \quad 3 \times (m \times m \times m \times m) \quad \text{that is} \quad 3 \times m \times m \times m \times m$$

(b) Here the bracketed expression is raised to the power 4 and so should be multiplied by itself four times:

$$(3m)^4 = (3m) \times (3m) \times (3m) \times (3m)$$

Because of the associativity of multiplication we can write this as

$$3 \times 3 \times 3 \times 3 \times m \times m \times m \times m \qquad \text{or simply} \quad 81m^4.$$

Note the important distinction between $(3m)^4$ and $3m^4$.

Exercises

1. Evaluate, without using a calculator, (a) 3^3, (b) 3^5, (c) 2^5. (d) 0.2^2, (e) 15^2.
2. Evaluate using a calculator (a) 7^3, (b) $(14)^{3.2}$.
3. Write each of the following using index notation:

   (a) $7 \times 7 \times 7 \times 7 \times 7$, (b) $t \times t \times t \times t$, (c) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} \times \frac{1}{7} \times \frac{1}{7}$.
4. Evaluate without using a calculator. Leave any fractions in fractional form.

   (a) $\left(\frac{2}{3}\right)^2$, (b) $\left(\frac{2}{5}\right)^3$, (c) $\left(\frac{1}{2}\right)^2$, (d) $\left(\frac{1}{2}\right)^3$, (e) 0.1^3.

Answers

1. (a) 27, (b) 243, (c) 32, (d) 0.04, (e) 225
2. (a) 343, (b) 4651.7 (1 d.p.).
3. (a) 7^5, (b) t^4, (c) $\left(\frac{1}{2}\right)^2\left(\frac{1}{7}\right)^3$
4. (a) $\dfrac{4}{9}$, (b) $\dfrac{8}{125}$, (c) $\dfrac{1}{4}$, (d) $\dfrac{1}{8}$, (e) 0.1^3 means $(0.1)\times(0.1)\times(0.1) = 0.001$

2. Laws of indices

There is a set of rules which enable us to manipulate expressions involving indices. These rules are known as the **laws of indices**, and they occur so commonly that it is worthwhile to memorise them.

Key Point 5

Laws of Indices

The laws of indices state:

First law: $a^m \times a^n = a^{m+n}$ add indices when multiplying numbers with the same base

Second law: $\dfrac{a^m}{a^n} = a^{m-n}$ subtract indices when dividing numbers with the same base

Third law: $(a^m)^n = a^{mn}$ multiply indices together when raising a number to a power

Example 16

Simplify (a) $a^5 \times a^4$, (b) $2x^5(x^3)$.

Solution

In each case we are required to multiply expressions involving indices. The bases are the same and we use the first law of indices.

(a) The indices must be added, thus $a^5 \times a^4 = a^{5+4} = a^9$.

(b) Because of the associativity of multiplication we can write

$$2x^5(x^3) = 2(x^5x^3) = 2x^{5+3} = 2x^8$$

The first law of indices (Key Point 5) extends in an obvious way when more terms are involved:

Example 17

Simplify $b^5 \times b^4 \times b^7$.

Solution

The indices are added. Thus $b^5 \times b^4 \times b^7 = b^{5+4+7} = b^{16}$.

Task

Simplify $y^4y^2y^3$.

Your solution

$y^4y^2y^3 =$

Answer

All quantities have the same base. To multiply the quantities together, the indices are added: y^9

Example 18

Simplify (a) $\dfrac{8^4}{8^2}$, (b) $x^{18} \div x^7$.

Solution

In each case we are required to divide expressions involving indices. The bases are the same and we use the second law of indices (Key Point 5).

(a) The indices must be subtracted, thus $\dfrac{8^4}{8^2} = 8^{4-2} = 8^2 = 64$.

(b) Again the indices are subtracted, and so $x^{18} \div x^7 = x^{18-7} = x^{11}$.

Task

Simplify $\dfrac{5^9}{5^7}$.

Your solution

$\dfrac{5^9}{5^7} =$

Answer

The bases are the same, and the division is carried out by subtracting the indices: $5^{9-7} = 5^2 = 25$

Task

Simplify $\dfrac{y^5}{y^2}$

Your solution

$\dfrac{y^5}{y^2} =$

Answer

$y^{5-2} = y^3$

Example 19

Simplify (a) $(8^2)^3$, (b) $(z^3)^4$.

Solution

We use the third law of indices (Key Point 5).

(a) $(8^2)^3 = 8^{2\times 3} = 8^6$

(b) $(z^3)^4 = z^{3\times 4} = z^{12}$.

Task

Simplify $(x^2)^5$.

Your solution

$(x^2)^5 =$

Answer

$x^{2\times 5} = x^{10}$

Task

Simplify $(e^x)^y$

Your solution

$(e^x)^y =$

Answer

Again, using the third law of indices, the two powers are multiplied: $e^{x\times y} = e^{xy}$

Two important results which can be derived from the laws of indices state:

Key Point 6

Any non-zero number raised to the power 0 has the value 1, that is $a^0 = 1$

Any number raised to power 1 is itself, that is $a^1 = a$

A generalisation of the third law of indices states:

Key Point 7

$$(a^m b^n)^k = a^{mk} b^{nk}$$

Example 20

Remove the brackets from (a) $(3x)^2$, (b) $(x^3y^7)^4$.

Solution

(a) Noting that $3 = 3^1$ and $x = x^1$ then $(3x)^2 = (3^1x^1)^2 = 3^2x^2 = 9x^2$

or, alternatively $(3x)^2 = (3x) \times (3x) = 9x^2$

(b) $(x^3y^7)^4 = x^{3\times 4}y^{7\times 4} = x^{12}y^{28}$

Exercises

1. Show that $(-xy)^2$ is equivalent to x^2y^2 whereas $(-xy)^3$ is equivalent to $-x^3y^3$.
2. Write each of the following expressions with a single index:

   (a) $6^7 6^9$, (b) $\dfrac{6^7}{6^{19}}$, (c) $(x^4)^3$
3. Remove the brackets from (a) $(8a)^2$, (b) $(7ab)^3$, (c) $7(ab)^3$, (d) $(6xy)^4$,
4. Simplify (a) $15x^2(x^3)$, (b) $3x^2(5x)$, (c) $18x^{-1}(3x^4)$.
5. Simplify (a) $5x(x^3)$, (b) $4x^2(x^3)$, (c) $3x^7(x^4)$, (d) $2x^8(x^{11})$, (e) $5x^2(3x^9)$

Answers

2. (a) 6^{16}, (b) 6^{-12}, (c) x^{12}
3. (a) $64a^2$, (b) $343a^3b^3$, (c) $7a^3b^3$, (d) $1296x^4y^4$
4. (a) $15x^5$, (b) $15x^3$, (c) $54x^3$
5. (a) $5x^4$, (b) $4x^5$, (c) $3x^{11}$, (d) $2x^{19}$, (e) $15x^{11}$

3. Polynomial expressions

An important group of mathematical expressions which use indices are known as **polynomials**. Examples of polynomials are

$$4x^3 + 2x^2 + 3x - 7, \qquad x^2 + x, \qquad 17 - 2t + 7t^4, \qquad z - z^3$$

Notice that they are all constructed using non-negative whole number powers of the variable. Recall that $x^0 = 1$ and so the number -7 appearing in the first expression can be thought of as $-7x^0$. Similarly the 17 appearing in the third expression can be read as $17t^0$.

Key Point 8

Polynomials

A polynomial expression takes the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$$

where a_0, a_1, a_2, a_3, $\ldots$ a_n are all constants called the **coefficients** of the polynomial. The number a_0 is also called the **constant term**. The highest power in a polynomial is called the **degree** of the polynomial.

Polynomials with low degrees have special names and subscript notation is often not needed:

Polynomial	**Degree**	**Name**
$ax^3 + bx^2 + cx + d$	3	cubic
$ax^2 + bx + c$	2	quadratic
$ax + b$	1	linear
a	0	constant

Task

Which of the following expressions are polynomials? Give the degree of those which are.

(a) $3x^2 + 4x + 2$, (b) $\dfrac{1}{x+1}$, (c) $\sqrt{x}$, (d) $2t + 4$,

(e) $3x^2 + \dfrac{4}{x} + 2$.

Recall that a polynomial expression must contain only terms involving non-negative whole number powers of the variable.

Give your answers by ringing the correct word (yes/no) and stating the degree if it is a polynomial.

Your solution

	polynomial		degree
(a) $3x^2 + 4x + 2$	yes	no	
(b) $\dfrac{1}{x+1}$	yes	no	
(c) $\sqrt{x}$	yes	no	
(d) $2t + 4$	yes	no	
(e) $3x^2 + \dfrac{4}{x} + 2$	yes	no	

Answer

(a) yes: polynomial of degree 2, called quadratic (b) no (c) no

(d) yes: polynomial of degree 1, called linear (e) no

Exercises

1. State which of the following are linear polynomials, which are quadratic polynomials, and which are constants.

   (a) x, (b) $x^2 + x + 3$, (c) $x^2 - 1$, (d) $3 - x$, (e) $7x - 2$, (f) $\frac{1}{2}$,

   (g) $\frac{1}{2}x + \frac{3}{4}$, (h) $3 - \frac{1}{2}x^2$.

2. State which of the following are polynomials.

   (a) $-\alpha^2 - \alpha - 1$, (b) $x^{1/2} - 7x^2$, (c) $\dfrac{1}{x}$, (d) 19.

3. Which of the following are polynomials ?

   (a) $4t + 17$, (b) $\dfrac{1}{2} - \dfrac{1}{2}t$, (c) 15, (d) $t^2 - 3t + 7$, (e) $\dfrac{1}{t^2} + \dfrac{1}{t} + 7$

4. State the degree of each of the following polynomials. For those of low degree, give their name.

   (a) $2t^3 + 7t^2$, (b) $7t^7 + 14t^3 - 2t^2$, (c) $7x + 2$,

   (d) $x^2 + 3x + 2$, (e) $2 - 3x - x^2$, (f) 42

Answers

1. (a), (d), (e) and (g) are linear. (b), (c) and (h) are quadratic. (f) is a constant.
2. (a) is a polynomial, (d) is a polynomial of degree 0. (b) and (c) are not polynomials.
3. (a) (b) (c) and (d) are polynomials.
4. (a) 3, cubic, (b) 7, (c) 1, linear, (d) 2, quadratic, (e) 2, quadratic, (f) 0, constant.

4. Negative indices

Sometimes a number is raised to a negative power. This is interpreted as follows:

Key Point 9

Negative Powers

$$a^{-m} = \frac{1}{a^m}, \qquad a^m = \frac{1}{a^{-m}}$$

Thus a negative index can be used to indicate a reciprocal.

Example 21

Write each of the following expressions using a positive index and simplify if possible.

(a) 2^{-3}, (b) $\frac{1}{4^{-3}}$, (c) x^{-1}, (d) x^{-2}, (e) 10^{-1}

Solution

(a) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, (b) $\frac{1}{4^{-3}} = 4^3 = 64$, (c) $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$, (d) $x^{-2} = \frac{1}{x^2}$,

(e) $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$ or 0.1.

Task

Write each of the following using a positive index. Use Key Point 9.

(a) $\frac{1}{t^{-4}}$, (b) 17^{-3}, (c) y^{-1}, (d) 10^{-2}

Your solution

(a) $\frac{1}{t^{-4}} =$

Answer

t^4

Your solution

(b) $17^{-3} =$

Answer

$\dfrac{1}{17^3}$

Your solution

(c) $y^{-1} =$

Answer

$\dfrac{1}{y}$

Your solution

(d) $10^{-2} =$

Answer

$\dfrac{1}{10^2}$ which equals $\dfrac{1}{100}$ or 0.01

Task

Simplify $\dfrac{a^8 \times a^7}{a^4}$

Use the first law of indices to simplify the numerator:

Your solution

$\dfrac{a^8 \times a^7}{a^4} =$

Answer

$\dfrac{a^{15}}{a^4}$

Now use the second law to simplify the result:

Your solution

Answer

a^{11}

Task

Simplify $\dfrac{m^9 \times m^{-2}}{m^{-3}}$

First simplify the numerator using the first law of indices:

Your solution

$\dfrac{m^9 \times m^{-2}}{m^{-3}} =$

Answer

$\dfrac{m^7}{m^{-3}}$

Then use the second law to simplify the result:

Your solution

Answer

$m^{7-(-3)} = m^{10}$

Exercises

1. Write the following numbers using a positive index and also express your answers as decimal fractions:

   (a) 10^{-1}, (b) 10^{-3}, (c) 10^{-4}

2. Simplify as much as possible:

   (a) x^3x^{-2}, (b) $\dfrac{t^4}{t^{-3}}$, (c) $\dfrac{y^{-2}}{y^{-6}}$.

Answers

1. (a) $\frac{1}{10} = 0.1$, (b) $\frac{1}{10^3} = 0.001$, (c) $\frac{1}{10^4} = 0.0001$.
2. (a) $x^1 = x$, (b) $t^{4+3} = t^7$, (c) $y^{-2+6} = y^4$.

5. Fractional indices

So far we have used indices that are whole numbers. We now consider fractional powers. Consider the expression $(16^{\frac{1}{2}})^2$. Using the third law of indices, $(a^m)^n = a^{mn}$, we can write

$$(16^{\frac{1}{2}})^2 = 16^{\frac{1}{2}\times 2} = 16^1 = 16$$

So $16^{\frac{1}{2}}$ is a number which when squared equals 16, that is 4 or -4. In other words $16^{\frac{1}{2}}$ is a square root of 16. There are always two square roots of a non-zero positive number, and we write $16^{\frac{1}{2}} = \pm 4$

Key Point 10

In general $\quad a^{\frac{1}{2}} \quad$ is a square root of $a \quad a \geq 0$

Similarly

$$(8^{\frac{1}{3}})^3 = 8^{\frac{1}{3}\times 3} = 8^1 = 8$$

so that $8^{\frac{1}{3}}$ is a number which when cubed equals 8. Thus $8^{\frac{1}{3}}$ is the cube root of 8, that is $\sqrt[3]{8}$, namely 2. Each number has only one cube root, and so

$$8^{\frac{1}{3}} = 2$$

In general

Key Point 11

$a^{\frac{1}{3}} \quad$ is the cube root of a

More generally we have

Key Point 12

The nth root of a is denoted by $a^{\frac{1}{n}}$.

When $a < 0$ the nth root only exists if n is odd.

If $a > 0$ the positive nth root is denoted by $\sqrt[n]{a}$

If $a < 0$ the negative nth root is $-\sqrt[n]{|a|}$

Your calculator will be able to evaluate fractional powers, and roots of numbers. Check that you can obtain the results of the following Examples on your calculator, but be aware that calculators normally give only one root when there may be others.

Example 22

Evaluate (a) $144^{1/2}$, (b) $125^{1/3}$

Solution

(a) $144^{1/2}$ is a square root of 144, that is ± 12.

(b) Noting that $5^3 = 125$, we see that $125^{1/3} = \sqrt[3]{125} = 5$

Example 23

Evaluate (a) $32^{1/5}$, (b) $32^{2/5}$, (c) $8^{2/3}$.

Solution

(a) $32^{\frac{1}{5}}$ is the 5th root of 32, that is $\sqrt[5]{32}$. Now $2^5 = 32$ and so $\sqrt[5]{32} = 2$.

(b) Using the third law of indices we can write $32^{2/5} = 32^{2\times\frac{1}{5}} = (32^{\frac{1}{5}})^2$. Thus

$$32^{2/5} = ((32)^{1/5})^2 = 2^2 = 4$$

(c) Note that $8^{1/3} = 2$. Then

$$8^{\frac{2}{3}} = 8^{2\times\frac{1}{3}} = (8^{1/3})^2 = 2^2 = 4$$

Note the following alternatives:

$$8^{2/3} = (8^{1/3})^2 = (8^2)^{1/3}$$

Example 24

Write the following as a simple power with a single index:

(a) $\sqrt{x^5}$, (b) $\sqrt[4]{x^3}$.

Solution

(a) $\sqrt{x^5} = (x^5)^{\frac{1}{2}}$. Then using the third law of indices we can write this as $x^{5\times\frac{1}{2}} = x^{\frac{5}{2}}$.

(b) $\sqrt[4]{x^3} = (x^3)^{\frac{1}{4}}$. Using the third law we can write this as $x^{3\times\frac{1}{4}} = x^{\frac{3}{4}}$.

Example 25

Show that $z^{-1/2} = \dfrac{1}{\sqrt{z}}$.

Solution

$$z^{-1/2} = \frac{1}{z^{1/2}} = \frac{1}{\sqrt{z}}$$

Task

Simplify $\dfrac{\sqrt{z}}{z^3 z^{-1/2}}$

First, rewrite $\sqrt{z}$ using an index and simplify the denominator using the first law of indices:

Your solution

$\dfrac{\sqrt{z}}{z^3 z^{-1/2}} =$

Answer

$\dfrac{z^{\frac{1}{2}}}{z^{\frac{5}{2}}}$

Finally, use the second law to simplify the result:

Your solution

Answer

$z^{\frac{1}{2}-\frac{5}{2}} = z^{-2}$ or $\dfrac{1}{z^2}$

Example 26

The generalisation of the third law of indices states that $(a^m b^n)^k = a^{mk} b^{nk}$. By taking $m = 1$, $n = 1$ and $k = \dfrac{1}{2}$ show that $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Solution

Taking $m = 1$, $n = 1$ and $k = \dfrac{1}{2}$ gives $(ab)^{1/2} = a^{1/2} b^{1/2}$.

Taking the case when all these roots are positive, we have $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Key Point 13

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \qquad a \geq 0, \quad b \geq 0$$

This result often allows answers to be written in alternative forms. For example, we may write $\sqrt{48}$ as $\sqrt{3 \times 16} = \sqrt{3}\sqrt{16} = 4\sqrt{3}$.

Although this rule works for multiplication we should be aware that it does **not** work for addition or subtraction so that

$$\sqrt{a \pm b} \neq \sqrt{a} \pm \sqrt{b}$$

Exercises

1. Evaluate using a calculator (a) $3^{1/2}$, (b) $15^{-\frac{1}{3}}$, (c) 85^3, (d) $81^{1/4}$
2. Evaluate using a calculator (a) 15^{-5}, (b) $15^{-2/7}$
3. Simplify (a) $\dfrac{a^{11}a^{3/4}}{a^{-1/2}}$, (b) $\dfrac{\sqrt{z}}{z^{3/2}}$, (c) $\dfrac{z^{-5/2}}{\sqrt{z}}$, (d) $\dfrac{\sqrt[3]{a}}{\sqrt[2]{a}}$, (e) $\dfrac{\sqrt[5]{z}}{z^{1/2}}$.
4. Write each of the following expressions with a single index:

   (a) $(x^{-4})^3$, (b) $x^{1/2}x^{1/4}$, (c) $\dfrac{x^{1/2}}{x^{1/4}}$

Answers

1 (a) 1.7321, (b) 0.4055, (c) 614125, (d) 3
2 (a) 0.000001317 (4 s.f.), (b) 0.4613 (4 s.f.),
3 (a) $a^{12.25}$, (b) z^{-1}, (c) z^{-3}, (d) $a^{-1/6}$, (e) $z^{-3/10}$
4 (a) x^{-12}, (b) $x^{3/4}$, (c) $x^{1/4}$

6. Scientific notation

It is often necessary to use very large or very small numbers such as 78000000 and 0.00000034. **Scientific notation** can be used to express such numbers in a more concise form. Each number is written in the form

$$a \times 10^n$$

where a is a number between 1 and 10. We can make use of the following facts:

$$10 = 10^1, \quad 100 = 10^2, \quad 1000 = 10^3 \qquad \text{and so on}$$

and

$$0.1 = 10^{-1}, \quad 0.01 = 10^{-2}, \quad 0.001 = 10^{-3} \qquad \text{and so on.}$$

For example,

- the number 5000 can be written $5 \times 1000 = 5 \times 10^3$
- the number 403 can be written $4.03 \times 100 = 4.03 \times 10^2$
- the number 0.009 can be written $9 \times 0.001 = 9 \times 10^{-3}$

Furthermore, to multiply a number by 10^n the decimal point is moved n places to the right if n is a positive integer, and n places to the left if n is a negative integer. (If necessary additional zeros are inserted to make up the required number of digits before the decimal point.)

Task

Write the numbers 0.00678 and 123456.7 in scientific notation.

Your solution

Answer

$0.00678 = 6.78 \times 10^{-3} \qquad 123456.7 = 1.234567 \times 10^5$

Engineering constants

Many constants appearing in engineering calculations are expressed in scientific notation. For example the charge on an electron equals 1.6×10^{-19} coulomb and the speed of light is 3×10^8 m s^{-1}. Avogadro's constant is equal to 6.023×10^{26} and is the number of atoms in one kilomole of an element. Clearly the use of scientific notation avoids writing lengthy strings of zeros.

Your scientific calculator will be able to accept numbers in scientific notation. Often the E button is used and a number like 4.2×10^7 will be entered as $4.2E7$. Note that $10E4$ means 10×10^4, that is 10^5. To enter the number 10^3 say, you would key in $1E3$. Entering powers of 10 incorrectly is a common cause of error. You must check how your particular calculator accepts numbers in scientific notation.

The following Task is designed to check that you can enter numbers given in scientific notation into your calculator.

Task

Use your calculator to find $4.2 \times 10^{-3} \times 3.6 \times 10^{-4}$.

Your solution

$4.2 \times 10^{-3} \times 3.6 \times 10^{-4} =$

Answer

1.512×10^{-6}

Exercises

1. Express each of the following numbers in scientific notation:

   (a) 45, (b) 456, (c) 2079, (d) 7000000, (e) 0.1, (f) 0.034,

   (g) 0.09856

2. Simplify $6 \times 10^{24} \times 1.3 \times 10^{-16}$

Answers

1. (a) 4.5×10^1, (b) 4.56×10^2, (c) 2.079×10^3, (d) 7×10^6, (e) 1×10^{-1},

   (f) 3.4×10^{-2}, (g) 9.856×10^{-2}

2. 7.8×10^8

Simplification and Factorisation

1.3

Introduction

In this Section we explain what is meant by the phrase 'like terms' and show how like terms are collected together and simplified.

Next we consider removing brackets. In order to simplify an expression which contains brackets it is often necessary to rewrite the expression in an equivalent form but without any brackets. This process of removing brackets must be carried out according to particular rules which are described in this Section.

Finally, factorisation, which can be considered as the reverse of the process, is dealt with. It is essential that you have had plenty practice in removing brackets before you study factorisation.

Prerequisites

Before starting this Section you should . . .

- be familiar with algebraic notation
- have competence in removing brackets

Learning Outcomes

On completion you should be able to . . .

- use the laws of indices
- simplify expressions by collecting like terms
- use the laws of indices
- identify common factors in an expression
- factorise simple expressions
- factorise quadratic expressions

1. Addition and subtraction of like terms

Like terms are multiples of the same quantity. For example $5y$, $17y$ and $\frac{1}{2}y$ are all multiples of y and so are like terms. Similarly, $3x^2$, $-5x^2$ and $\frac{1}{4}x^2$ are all multiples of x^2 and so are like terms.

Further examples of like terms are:

kx and ℓx which are both multiples of x,

x^2y, $6x^2y$, $-13x^2y$, $-2yx^2$, which are all multiples of x^2y

abc^2, $-7abc^2$, $kabc^2$, are all multiples of abc^2

Like terms can be added or subtracted in order to simplify expressions.

Example 27

Simplify $5x - 13x + 22x$.

Solution

All three terms are multiples of x and so are like terms. The expression can be simplified to $14x$.

Example 28

Simplify $5z + 2x$.

Solution

$5z$ and $2x$ are not like terms. They are not multiples of the same quantity. This expression cannot be simplified.

Task

Simplify $5a + 2b - 7a - 9b$.

Your solution

$5a + 2b - 7a - 9b =$

Answer

$-2a - 7b$

Example 29

Simplify $2x^2 - 7x + 11x^2 + x$.

Solution

$2x^2$ and $11x^2$, both being multiples of x^2, can be collected together and added to give $13x^2$.

Similarly, $-7x$ and x can be added to give $-6x$.

We get $2x^2 - 7x + 11x^2 + x = 13x^2 - 6x$ which cannot be simplified further.

Task

Simplify $\frac{1}{2}x + \frac{3}{4}x - 2y$.

Your solution

$\frac{1}{2}x + \frac{3}{4}x - 2y =$

Answer

$\frac{5}{4}x - 2y$

Example 30

Simplify $3a^2b - 7a^2b - 2b^2 + a^2$.

Solution

Note that $3a^2b$ and $7a^2b$ are both multiples of a^2b and so are like terms. There are no other like terms. Therefore

$$3a^2b - 7a^2b - 2b^2 + a^2 = -4a^2b - 2b^2 + a^2$$

Exercises

1. Simplify, if possible,

   (a) $5x + 2x + 3x$, (b) $3q - 2q + 11q$, (c) $7x^2 + 11x^2$, (d) $-11v^2 + 2v^2$, (e) $5p + 3q$

2. Simplify, if possible, (a) $5w + 3r - 2w + r$, (b) $5w^2 + w + 1$, (c) $6w^2 + w^2 - 3w^2$

3. Simplify, if possible,

   (a) $7x + 2 + 3x + 8x - 11$, (b) $2x^2 - 3x + 6x - 2$, (c) $-5x^2 - 3x^2 + 11x + 11$,

   (d) $4q^2 - 4r^2 + 11r + 6q$, (e) $a^2 + ba + ab + b^2$, (f) $3x^2 + 4x + 6x + 8$,

   (g) $s^3 + 3s^2 + 2s^2 + 6s + 4s + 12$.

4. Explain the distinction, if any, between each of the following expressions, and simplify if possible.

   (a) $18x - 9x$, (b) $18x(9x)$, (c) $18x(-9x)$, (d) $-18x - 9x$, (e) $-18x(9x)$

5. Explain the distinction, if any, between each of the following expressions, and simplify if possible.

   (a) $4x - 2x$, (b) $4x(-2x)$, (c) $4x(2x)$, (d) $-4x(2x)$, (e) $-4x - 2x$, (f) $(4x)(2x)$

6. Simplify, if possible,

   (a) $\dfrac{2}{3}x^2 + \dfrac{x^2}{3}$, (b) $0.5x^2 + \frac{3}{4}x^2 - \frac{11}{2}x$, (c) $3x^3 - 11x + 3yx + 11$,

   (d) $-4\alpha x^2 + \beta x^2$ where α and β are constants.

Answers

1. (a) $10x$, (b) $12q$, (c) $18x^2$, (d) $-9v^2$, (e) cannot be simplified.
2. (a) $3w + 4r$, (b) cannot be simplified, (c) $4w^2$
3. (a) $18x - 9$, (b) $2x^2 + 3x - 2$, (c) $-8x^2 + 11x + 11$, (d) cannot be simplified,

   (e) $a^2 + 2ab + b^2$, (f) $3x^2 + 10x + 8$, (g) $s^3 + 5s^2 + 10s + 12$
4. (a) $9x$, (b) $162x^2$, (c) $-162x^2$, (d) $-27x$, (e) $-162x^2$
5. (a) $4x - 2x = 2x$, (b) $4x(-2x) = -8x^2$, (c) $4x(2x) = 8x^2$, (d) $-4x(2x) = -8x^2$,

   (e) $-4x - 2x = -6x$, (f) $(4x)(2x) = 8x^2$
6. (a) x^2, (b) $1.25x^2 - \dfrac{11}{2}x$, (c) cannot be simplified, (d) $(\beta - 4\alpha)x^2$

2. Removing brackets from expressions $a(b+c)$ and $a(b-c)$

Removing brackets means **multiplying out**. For example $5(2+4) = 5 \times 2 + 5 \times 4 = 10 + 20 = 30$. In this simple example we could alternatively get the same result as follows: $5(2+4) = 5 \times 6 = 30$. That is:

$$5(2+4) = 5 \times 2 + 5 \times 4$$

In an expression such as $5(x+y)$ it is intended that the 5 multiplies both x and y to produce $5x+5y$. Thus the expressions $5(x+y)$ and $5x+5y$ are equivalent. In general we have the following rules known as **distributive laws**:

Key Point 14

$$a(b+c) = ab + ac$$

$$a(b-c) = ab - ac$$

Note that when the brackets are removed both terms in the brackets are multiplied by a.

As we have noted above, if you insert numbers instead of letters into these expressions you will see that both left and right hand sides are equivalent. For example

$4(3+5)$ has the same value as $4(3) + 4(5)$, that is 32

and

$7(8-3)$ has the same value as $7(8) - 7(3)$, that is 35

Example 31

Remove the brackets from (a) $9(2+y)$, (b) $9(2y)$.

Solution

(a) In the expression $9(2+y)$ the 9 must multiply both terms in the brackets:

$$\begin{aligned} 9(2+y) &= 9(2) + 9(y) \\ &= 18 + 9y \end{aligned}$$

(b) Recall that $9(2y)$ means $9 \times (2 \times y)$ and that when multiplying numbers together the presence of brackets is irrelevant. Thus $9(2y) = 9 \times 2 \times y = 18y$

The crucial distinction between the role of the factor 9 in the two expressions $9(2 + y)$ and $9(2y)$ in Example 31 should be noted.

Example 32

Remove the brackets from $9(x + 2y)$.

Solution

In the expression $9(x + 2y)$ the 9 must multiply both the x and the $2y$ in the brackets. Thus

$$\begin{aligned} 9(x + 2y) &= 9x + 9(2y) \\ &= 9x + 18y \end{aligned}$$

Task

Remove the brackets from $9(2x + 3y)$.

Remember that the 9 must multiply both the term $2x$ and the term $3y$:

Your solution

$9(2x + 3y) =$

Answer

$18x + 27y$

Example 33

Remove the brackets from $-3(5x - z)$.

Solution

The number -3 must multiply both the $5x$ and the z.

$$\begin{aligned} -3(5x - z) &= (-3)(5x) - (-3)(z) \\ &= -15x + 3z \end{aligned}$$

Task

Remove the brackets from $6x(3x - 2y)$.

Your solution

Answer

$$6x(3x - 2y) = 6x(3x) - 6x(2y) = 18x^2 - 12xy$$

Example 34

Remove the brackets from $-(3x + 1)$.

Solution

Although the 1 is unwritten, the minus sign outside the brackets stands for -1. We must therefore consider the expression $-1(3x + 1)$.

$$\begin{aligned} -1(3x+1) &= (-1)(3x) + (-1)(1) \\ &= -3x + (-1) \\ &= -3x - 1 \end{aligned}$$

Task

Remove the brackets from $-(5x - 3y)$.

Your solution

Answer

$-(5x - 3y)$ means $-1(5x - 3y)$.

$$-1(5x - 3y) = (-1)(5x) - (-1)(3y) = -5x + 3y$$

Task

Remove the brackets from $m(m-n)$.

In the expression $m(m-n)$ the first m must multiply both terms in the brackets:

Your solution

$m(m-n) =$

Answer

$m^2 - mn$

Example 35

Remove the brackets from the expression $5x - (3x+1)$ and simplify the result by collecting like terms.

Solution

The brackets in $-(3x+1)$ were removed in Example 34 on page 46.

$$\begin{aligned} 5x - (3x+1) &= 5x - 1(3x+1) \\ &= 5x - 3x - 1 \\ &= 2x - 1 \end{aligned}$$

Example 36

Show that $\dfrac{-x-1}{4}$, $\dfrac{-(x+1)}{4}$ and $-\dfrac{x+1}{4}$ are all equivalent expressions.

Solution

Consider $-(x+1)$. Removing the brackets we obtain $-x-1$ and so

$$\frac{-x-1}{4} \quad \text{is equivalent to} \quad \frac{-(x+1)}{4}$$

A negative quantity divided by a positive quantity will be negative. Hence

$$\frac{-(x+1)}{4} \quad \text{is equivalent to} \quad -\frac{x+1}{4}$$

You should study all three expressions carefully to recognise the variety of equivalent ways in which we can write an algebraic expression.

Sometimes the bracketed expression can appear on the left, as in $(a + b)c$. To remove the brackets here we use the following rules:

Key Point 15

$$(a + b)c = ac + bc$$

$$(a - b)c = ac - bc$$

Note that when the brackets are removed both the terms in the brackets multiply c.

Example 37

Remove the brackets from $(2x + 3y)x$.

Solution

Both terms in the brackets multiply the x outside. Thus

$$\begin{aligned}(2x + 3y)x &= 2x(x) + 3y(x) \\ &= 2x^2 + 3yx\end{aligned}$$

Task

Remove the brackets from (a) $(x + 3)(-2)$, (b) $(x - 3)(-2)$.

Your solution

(a) $(x + 3)(-2) =$

Answer

Both terms in the bracket must multiply the -2, giving $-2x - 6$

Your solution

(b) $(x - 3)(-2) =$

Answer

$-2x + 6$

3. Removing brackets from expressions of the form $(a+b)(c+d)$

Sometimes it is necessary to consider two bracketed terms multiplied together. In the expression $(a+b)(c+d)$, by regarding the first bracket as a single term we can use the result in Key Point 14 to write it as $(a+b)c+(a+b)d$. Removing the brackets from each of these terms produces $ac+bc+ad+bd$. More concisely:

Key Point 16

$$(a+b)(c+d) = (a+b)c+(a+b)d = ac+bc+ad+bd$$

We see that each term in the first bracketed expression multiplies each term in the second bracketed expression.

Example 38

Remove the brackets from $(3+x)(2+y)$

Solution

We find
$$\begin{aligned}(3+x)(2+y) &= (3+x)(2)+(3+x)y\\ &= (3)(2)+(x)(2)+(3)(y)+(x)(y) = 6+2x+3y+xy\end{aligned}$$

Example 39

Remove the brackets from $(3x+4)(x+2)$ and simplify your result.

Solution

$$\begin{aligned}(3x+4)(x+2) &= (3x+4)(x)+(3x+4)(2)\\ &= 3x^2+4x+6x+8 = 3x^2+10x+8\end{aligned}$$

Example 40

Remove the brackets from $(a+b)^2$ and simplify your result.

Solution

When a quantity is squared it is multiplied by itself. Thus

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= (a+b)a + (a+b)b \\ &= a^2 + ba + ab + b^2 = a^2 + 2ab + b^2\end{aligned}$$

Key Point 17

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Task

Remove the brackets from the following expressions and simplify the results.

(a) $(x+7)(x+3)$, (b) $(x+3)(x-2)$,

Your solution

(a) $(x+7)(x+3) =$

Answer

$x^2 + 7x + 3x + 21 = x^2 + 10x + 21$

Your solution

(b) $(x+3)(x-2) =$

Answer

$x^2 + 3x - 2x - 6 = x^2 + x - 6$

Example 41

Explain the distinction between $(x+3)(x+2)$ and $x+3(x+2)$.

Solution

In the first expression removing the brackets we find

$$\begin{aligned}(x+3)(x+2) &= x^2+3x+2x+6\\ &= x^2+5x+6\end{aligned}$$

In the second expression we have

$$x+3(x+2) = x+3x+6 = 4x+6$$

Note that in the second expression the term $(x+2)$ is only multiplied by 3 and not by x.

Example 42

Remove the brackets from $(s^2+2s+4)(s+3)$.

Solution

Each term in the first bracket must multiply each term in the second. Working through all combinations systematically we have

$$\begin{aligned}(s^2+2s+4)(s+3) &= (s^2+2s+4)(s) \quad + \quad (s^2+2s+4)(3)\\ &= s^3+2s^2+4s+3s^2+6s+12\\ &= s^3+5s^2+10s+12\end{aligned}$$

Engineering Example 1

Reliability in a communication network

Introduction

The reliability of a communication network depends on the reliability of its component parts. The reliability of a component can be represented by a number between 0 and 1 which represents the probability that it will function over a given period of time.

A very simple system with only two components C_1 and C_2 can be configured in series or in parallel. If the components are in **series** then the system will fail if one component fails (see Figure 4)

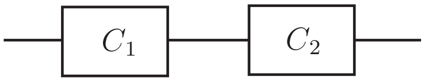


Figure 4: Both components 1 and 2 must function for the system to function

If the components are in **parallel** then only one component need function properly (see Figure 5) and we have built-in redundancy.

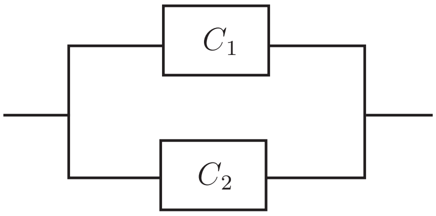


Figure 5: Either component 1 or 2 must function for the system to function

The reliability of a system with two units in parallel is given by $1-(1-R_1)(1-R_2)$ which is the same as $R_1 + R_2 - R_1R_2$, where R_i is the reliability of component C_i. The reliability of a system with 3 units in parallel, as in Figure 6, is given by

$$1-(1-R_1)(1-R_2)(1-R_3)$$

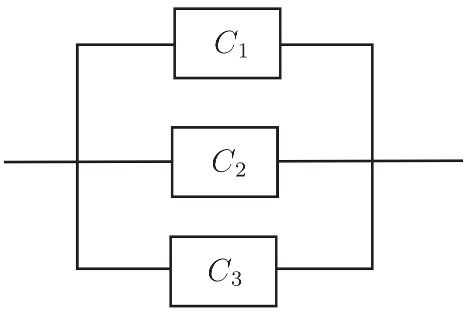


Figure 6: At least one of the three components must function for the system to function

Problem in words

(a) Show that the expression for the system reliability for three components in parallel is equal to $R_1 + R_2 + R_3 - R_1R_2 - R_1R_3 - R_2R_3 + R_1R_2R_3$

(b) Find an expression for the reliability of the system when the reliability of each of the components is the same i.e. $R_1 = R_2 = R_3 = R$

(c) Find the system reliability when $R = 0.75$

(d) Find the system reliability when there are two parallel components each with reliability $R = 0.75$.

Mathematical statement of the problem

(a) Show that $1-(1-R_1)(1-R_2)(1-R_3) \equiv R_1+R_2+R_3-R_1R_2-R_1R_3-R_2R_3+R_1R_2R_3$

(b) Find $1 - (1 - R_1)(1 - R_2)(1 - R_3)$ in terms of R when $R_1 = R_2 = R_3 = R$

(c) Find the value of (b) when $R = 0.75$

(d) Find $1 - (1 - R_1)(1 - R_2)$ when $R_1 = R_2 = 0.75$.

Mathematical analysis

(a)
$$\begin{aligned}
1 - (1 - R_1)(1 - R_2)(1 - R_3) &\equiv 1 - (1 - R_1 - R_2 + R_1R_2)(1 - R_3) \\
&= 1 - ((1 - R_1 - R_2 + R_1R_2) \times 1 - (1 - R_1 - R_2 + R_1R_2) \times R_3) \\
&= 1 - (1 - R_1 - R_2 + R_1R_2 - (R_3 - R_1R_3 - R_2R_3 + R_1R_2R_3)) \\
&= 1 - (1 - R_1 - R_2 + R_1R_2 - R_3 + R_1R_3 + R_2R_3 - R_1R_2R_3) \\
&= 1 - 1 + R_1 + R_2 - R_1R_2 + R_3 - R_1R_3 - R_2R_3 + R_1R_2R_3 \\
&= R_1 + R_2 + R_3 - R_1R_2 - R_1R_3 - R_2R_3 + R_1R_2R_3
\end{aligned}$$

(b) When $R_1 = R_2 = R_3 = R$ the reliability is

$1 - (1 - R)^3$ which is equivalent to $3R - 3R^2 + R^3$

(c) When $R_1 = R_2 = R_3 = 0.75$ we get

$$1 - (1 - 0.75)^3 = 1 - 0.25^3 = 1 - 0.015625 = 0.984375$$

(d) $1 - (0.25)^2 = 0.9375$

Interpretation

The mathematical analysis confirms the expectation that the more components there are in parallel then the more reliable the system becomes (1 component: 0.75; 2 components: 0.9375; 3 components: 0.984375). With three components in parallel, as in part (c), although each individual component is relatively unreliable ($R = 0.75$ implies a one in four chance of failure of an individual component) the system as a whole has an over 98% probability of functioning (under 1 in 50 chance of failure).

Exercises

1. Remove the brackets from each of the following expressions:

(a) $2(mn)$, (b) $2(m+n)$, (c) $a(mn)$, (d) $a(m+n)$, (e) $a(m-n)$,
(f) $(am)n$, (g) $(a+m)n$, (h) $(a-m)n$, (i) $5(pq)$, (j) $5(p+q)$,
(k) $5(p-q)$, (l) $7(xy)$, (m) $7(x+y)$, (n) $7(x-y)$, (o) $8(2p+q)$,
(p) $8(2pq)$, (q) $8(2p-q)$, (r) $5(p-3q)$, (s) $5(p+3q)$ (t) $5(3pq)$.

2. Remove the brackets from each of the following expressions:

(a) $4(a+b)$, (b) $2(m-n)$, (c) $9(x-y)$,

3. Remove the brackets from each of the following expressions and simplify where possible:

(a) $(2+a)(3+b)$, (b) $(x+1)(x+2)$, (c) $(x+3)(x+3)$, (d) $(x+5)(x-3)$

4. Remove the brackets from each of the following expressions:

(a) $(7+x)(2+x)$, (b) $(9+x)(2+x)$, (c) $(x+9)(x-2)$, (d) $(x+11)(x-7)$,
(e) $(x+2)x$, (f) $(3x+1)x$, (g) $(3x+1)(x+1)$, (h) $(3x+1)(2x+1)$,
(i) $(3x+5)(2x+7)$, (j) $(3x+5)(2x-1)$, (k) $(5-3x)(x+1)$ (l) $(2-x)(1-x)$.

5. Remove the brackets from $(s+1)(s+5)(s-3)$.

Answers

1. (a) $2mn$, (b) $2m+2n$, (c) amn, (d) $am+an$, (e) $am-an$, (f) amn, (g) $an+mn$,
(h) $an-mn$, (i) $5pq$, (j) $5p+5q$, (k) $5p-5q$, (l) $7xy$, (m) $7x+7y$, (n) $7x-7y$,
(o) $16p+8q$, (p) $16pq$, (q) $16p-8q$, (r) $5p-15q$, (s) $5p+15q$, (t) $15pq$

2. (a) $4a+4b$, (b) $2m-2n$, (c) $9x-9y$

3. (a) $6+3a+2b+ab$, (b) x^2+3x+2, (c) x^2+6x+9, (d) $x^2+2x-15$

4. On removing brackets we obtain:

(a) $14+9x+x^2$, (b) $18+11x+x^2$, (c) $x^2+7x-18$, (d) $x^2+4x-77$
(e) x^2+2x, (f) $3x^2+x$, (g) $3x^2+4x+1$ (h) $6x^2+5x+1$
(i) $6x^2+31x+35$, (j) $6x^2+7x-5$, (k) $-3x^2+2x+5$, (l) x^2-3x+2

5. $s^3+3s^2-13s-15$

4. Factorisation

A number is said to be **factorised** when it is written as a product. For example, 21 can be factorised into 7×3. We say that 7 and 3 are **factors** of 21.

Algebraic expressions can also be factorised. Consider the expression $7(2x + 1)$. Removing the brackets we can rewrite this as

$$7(2x + 1) = 7(2x) + (7)(1) = 14x + 7.$$

Thus $14x + 7$ is equivalent to $7(2x + 1)$. We see that $14x + 7$ has factors 7 and $(2x + 1)$. The factors 7 and $(2x + 1)$ *multiply* together to give $14x + 7$. The process of writing an expression as a product of its factors is called **factorisation**. When asked to factorise $14x + 7$ we write

$$14x + 7 = 7(2x + 1)$$

and so we see that factorisation can be regarded as reversing the process of removing brackets.

Always remember that the factors of an algebraic expression are *multiplied* together.

Example 43

Factorise the expression $4x + 20$.

Solution

Both terms in the expression $4x + 20$ are examined to see if they have any factors in common. Clearly 20 can be factorised as $(4)(5)$ and so we can write

$$4x + 20 = 4x + (4)(5)$$

The factor 4 is common to both terms on the right; it is called a **common factor** and is placed at the front and outside brackets to give

$$4x + 20 = 4(x + 5)$$

Note that the factorised form can be checked by removing the brackets again.

Example 44

Factorise $z^2 - 5z$.

Solution

Note that since $z^2 = z \times z$ we can write

$$z^2 - 5z = z(z) - 5z$$

so that there is a common factor of z. Hence

$$z^2 - 5z = z(z) - 5z = z(z - 5)$$

Example 45
Factorise $6x - 9y$.

Solution

By observation, we see that there is a common factor of 3. Thus $6x - 9y = 3(2x - 3y)$

Task
Factorise $14z + 21w$.

(a) Find the factor common to both $14z$ and $21w$:

Your solution

Answer
7

(b) Now factorise $14z + 21w$:

Your solution
$14z + 21w =$

Answer
$7(2z + 3w)$

Note: If you have any doubt, you can check your answer by removing the brackets again.

Task
Factorise $6x - 12xy$.

First identify the two common factors:

Your solution

Answer
6 and x

Now factorise $6x - 12xy$:

Your solution
$6x - 12xy =$

Answer
$6x(1 - 2y)$

Exercises

1. Factorise

   (a) $5x + 15y$, (b) $3x - 9y$, (c) $2x + 12y$, (d) $4x + 32z + 16y$, (e) $\frac{1}{2}x + \frac{1}{4}y$.

   In each case check your answer by removing the brackets again.

2. Factorise

   (a) $a^2 + 3ab$, (b) $xy + xyz$, (c) $9x^2 - 12x$

3. Explain why a is a factor of $a + ab$ but b is not. Factorise $a + ab$.

4. Explain why x^2 is a factor of $4x^2 + 3yx^3 + 5yx^4$ but y is not.

   Factorise $4x^2 + 3yx^3 + 5yx^4$.

Answers

1. (a) $5(x + 3y)$, (b) $3(x - 3y)$, (c) $2(x + 6y)$, (d) $4(x + 8z + 4y)$, (e) $\frac{1}{2}(x + \frac{1}{2}y)$
2. (a) $a(a + 3b)$, (b) $xy(1 + z)$, (c) $3x(3x - 4)$.
3. $a(1 + b)$.
4. $x^2(4 + 3yx + 5yx^2)$.

5. Factorising quadratic expressions

Quadratic expressions commonly occur in many areas of mathematics, physics and engineering. Many quadratic expressions can be written as the product of two linear factors and, in this Section, we examine how these factors can be easily found.

Key Point 18

An expression of the form

$$ax^2 + bx + c \qquad a \neq 0$$

where a, b and c are numbers is called a **quadratic expression** (in the variable x).

The numbers b and c may be zero but a must not be zero (for, then, the quadratic reduces to a linear expression or constant). The number a is called the **coefficient** of x^2, b is the coefficient of x and c is called the **constant term**.

Case 1

Consider the product $(x+1)(x+2)$. Removing brackets yields x^2+3x+2. Conversely, we see that the factors of x^2+3x+2 are $(x+1)$ and $(x+2)$. However, if we were given the quadratic expression first, how would we factorise it ? The following examples show how to do this but note that not all quadratic expressions can be easily factorised.

To enable us to factorise a quadratic expression in which the coefficient of x^2 equals 1, we note the following expansion:

$$(x+m)(x+n) = x^2 + mx + nx + mn = x^2 + (m+n)x + mn$$

So, given a quadratic expression we can think of the coefficient of x as $m+n$ and the constant term as mn. Once the values of m and n have been found the factors can be easily obtained.

Example 46

Factorise $x^2 + 4x - 5$.

Solution

Writing $x^2 + 4x - 5 = (x+m)(x+n) = x^2 + (m+n)x + mn$ we seek numbers m and n such that $m+n = 4$ and $mn = -5$. By trial and error it is not difficult to find that $m = 5$ and $n = -1$ (or, the other way round, $m = -1$ and $n = 5$). So we can write

$$x^2 + 4x - 5 = (x+5)(x-1)$$

The answer can be checked easily by removing brackets.

Task

Factorise $x^2 + 6x + 8$.

As the coefficient of x^2 is 1, we can write

$$x^2 + 6x + 8 = (x+m)(x+n) = x^2 + (m+n)x + mn$$

so that $m+n = 6$ and $mn = 8$.

First, find suitable values for m and n:

Your solution

Answer

$m = 4,\ n = 2$ or, the other way round, $m = 2,\ n = 4$

Finally factorise the quadratic:

Your solution

$x^2 + 6x + 8 =$

Answer

$(x+4)(x+2)$

Case 2

When the coefficient of x^2 is not equal to 1 it may be possible to extract a *numerical* factor. For example, note that $3x^2 + 18x + 24$ can be written as $3(x^2 + 6x + 8)$ and then factorised as in the previous Task in Case 1. Sometimes no numerical factor can be found and a slightly different approach may be taken. We will demonstrate a technique which can always be used to transform the given expression into one in which the coefficient of the squared variable equals 1.

Example 47

Factorise $2x^2 + 5x + 3$.

Solution

First note the coefficient of x^2; in this case 2. Multiply the whole expression by this number and rearrange as follows:

$$2(2x^2 + 5x + 3) = 2(2x^2) + 2(5x) + 2(3) = (2x)^2 + 5(2x) + 6.$$

We now introduce a new variable z such that $z = 2x$ Thus we can write

$$(2x)^2 + 5(2x) + 6 \qquad \text{as} \qquad z^2 + 5z + 6$$

This can be factorised to give $(z+3)(z+2)$. Returning to the original variable by replacing z by $2x$ we find

$$2(2x^2 + 5x + 3) = (2x+3)(2x+2)$$

A factor of 2 can be extracted from the second bracket on the right so that

$$2(2x^2 + 5x + 3) = 2(2x+3)(x+1)$$

so that

$$2x^2 + 5x + 3 = (2x+3)(x+1)$$

As an alternative to the technique of Example 47, experience and practice can often help us to identify factors. For example suppose we wish to factorise $3x^2 + 7x + 2$. We write

$$3x^2 + 7x + 2 = (\qquad\qquad)(\qquad\qquad)$$

In order to obtain the term $3x^2$ we can place terms $3x$ and x in the brackets to give

$$3x^2 + 7x + 2 = (3x + \quad ? \quad)(x + \quad ? \quad)$$

In order to obtain the constant 2, we consider the factors of 2. These are 1,2 or -1,-2. By placing these factors in the brackets we can factorise the quadratic expression. Various possibilities exist: we could write $(3x+2)(x+1)$ or $(3x+1)(x+2)$ or $(3x-2)(x-1)$ or $(3x-1)(x-2)$, only one of which is correct. By removing brackets from each in turn we look for the factorisation which produces the correct middle term, $7x$. The correct factorisation is found to be

$$3x^2 + 7x + 2 = (3x + 1)(x + 2)$$

With practice you will be able to carry out this process quite easily.

Task

Factorise the quadratic expression $5x^2 - 7x - 6$.

Write $5x^2 - 7x - 6 = (\qquad)(\qquad)$

To obtain the quadratic term $5x^2$, insert $5x$ and x in the brackets:

$$5x^2 - 7x - 6 = (5x + \ ?\ )(x + \ ?\ )$$

Now find the factors of -6:

Your solution

Answer

$3, -2$ or $-3, 2$ or $-6, 1$ or $6, -1$

Use these factors in turn to find which pair, if any, gives rise to the middle term, $-7x$, and complete the factorisation:

Your solution

$5x^2 - 7x - 6 = (5x + \ )(x + \ ) =$

Answer

$(5x + 3)(x - 2)$

On occasions you will meet expressions of the form $x^2 - y^2$ known as the **difference of two squares**. It is easy to verify by removing brackets that this factorises as

$$x^2 - y^2 = (x + y)(x - y)$$

So, if you can learn to recognise such expressions it is an easy matter to factorise them.

Example 48

Factorise

(a) $x^2 - 36z^2$, (b) $25x^2 - 9z^2$, (c) $\alpha^2 - 1$

Solution

In each case we are required to find the difference of two squared terms.

(a) Note that $x^2 - 36z^2 = x^2 - (6z)^2$. This factorises as $(x + 6z)(x - 6z)$.

(b) Here $25x^2 - 9z^2 = (5x)^2 - (3z)^2$. This factorises as $(5x + 3z)(5x - 3z)$.

(c) $\alpha^2 - 1 = (\alpha + 1)(\alpha - 1)$.

Exercises

1. Factorise

   (a) $x^2 + 8x + 7$, (b) $x^2 + 6x - 7$, (c) $x^2 + 7x + 10$, (d) $x^2 - 6x + 9$.

2. Factorise

   (a) $2x^2 + 3x + 1$, (b) $2x^2 + 4x + 2$, (c) $3x^2 - 3x - 6$, (d) $5x^2 - 4x - 1$, (e) $16x^2 - 1$,

   (f) $-x^2 + 1$, (g) $-2x^2 + x + 3$.

3. Factorise

   (a) $x^2 + 9x + 14$, (b) $x^2 + 11x + 18$, (c) $x^2 + 7x - 18$, (d) $x^2 + 4x - 77$,
   (e) $x^2 + 2x$, (f) $3x^2 + x$, (g) $3x^2 + 4x + 1$, (h) $6x^2 + 5x + 1$,
   (i) $6x^2 + 31x + 35$, (j) $6x^2 + 7x - 5$, (k) $-3x^2 + 2x + 5$, (l) $x^2 - 3x + 2$.

4. Factorise (a) $z^2 - 144$, (b) $z^2 - \frac{1}{4}$, (c) $s^2 - \frac{1}{9}$

Answers

1. (a) $(x + 7)(x + 1)$, (b) $(x + 7)(x - 1)$, (c) $(x + 2)(x + 5)$, (d) $(x - 3)(x - 3)$

2. (a) $(2x + 1)(x + 1)$, (b) $2(x + 1)^2$, (c) $3(x + 1)(x - 2)$, (d)$(5x + 1)(x - 1)$,

   (e) $(4x + 1)(4x - 1)$, (f) $(x + 1)(1 - x)$, (g) $(x + 1)(3 - 2x)$

3. The factors are:

   (a) $(7 + x)(2 + x)$, (b) $(9 + x)(2 + x)$, (c) $(x + 9)(x - 2)$, (d) $(x + 11)(x - 7)$,
   (e) $(x + 2)x$, (f) $(3x + 1)x$, (g) $(3x + 1)(x + 1)$, (h) $(3x + 1)(2x + 1)$,
   (i) $(3x + 5)(2x + 7)$, (j) $(3x + 5)(2x - 1)$, (k) $(5 - 3x)(x + 1)$, (l) $(2 - x)(1 - x)$.

4. (a) $(z + 12)(z - 12)$, (b) $(z + \frac{1}{2})(z - \frac{1}{2})$, (c) $(s + \frac{1}{3})(s - \frac{1}{3})$

Arithmetic of Algebraic Fractions

1.4

Introduction

Just as one whole number divided by another is called a numerical fraction, so one algebraic expression divided by another is known as an **algebraic fraction**. Examples are

$$\frac{x}{y}, \quad \frac{3x+2y}{x-y}, \quad \text{and} \quad \frac{x^2+3x+1}{x-4}$$

In this Section we explain how algebraic fractions can be simplified, added, subtracted, multiplied and divided.

Prerequisites

Before starting this Section you should ...

- be familiar with the arithmetic of numerical fractions

Learning Outcomes

On completion you should be able to ...

- add, subtract, multiply and divide algebraic fractions

1. Cancelling common factors

Consider the fraction $\frac{10}{35}$. To simplify it we can factorise the numerator and the denominator and then cancel any common factors. Common factors are those factors which occur in both the numerator and the denominator. Thus

$$\frac{10}{35} = \frac{\cancel{5} \times 2}{7 \times \cancel{5}} = \frac{2}{7}$$

Note that the common factor 5 has been cancelled. It is important to remember that only *common factors* can be cancelled. The fractions $\frac{10}{35}$ and $\frac{2}{7}$ have identical values - they are equivalent fractions - but $\frac{2}{7}$ is in a simpler form than $\frac{10}{35}$.

We apply the same process when simplifying algebraic fractions.

Example 49

Simplify, if possible,

(a) $\frac{yx}{2x}$, (b) $\frac{x}{xy}$, (c) $\frac{x}{x+y}$

Solution

(a) In the expression $\frac{yx}{2x}$, x is a factor common to both numerator and denominator. This common factor can be cancelled to give

$$\frac{y\,\cancel{x}}{2\,\cancel{x}} = \frac{y}{2}$$

(b) Note that $\frac{x}{xy}$ can be written $\frac{1x}{xy}$. The common factor of x can be cancelled to give

$$\frac{1\,\cancel{x}}{\cancel{x}y} = \frac{1}{y}$$

(c) In the expression $\frac{x}{x+y}$ notice that an x appears in both numerator and denominator. However x is not a common factor. Recall that factors of an expression are *multiplied* together whereas in the denominator x is *added* to y. This expression cannot be simplified.

Simplify, if possible, (a) $\dfrac{abc}{3ac}$, (b) $\dfrac{3ab}{b+a}$

When simplifying remember only common factors can be cancelled.

Your solution

(a) $\dfrac{abc}{3ac} =$ (b) $\dfrac{3ab}{b+a} =$

Answer

(a) $\dfrac{b}{3}$ (b) This cannot be simplified.

Simplify $\dfrac{21x^3}{14x}$,

Your solution

Answer

Factorising and cancelling common factors gives:

$$\frac{21x^3}{14x} = \frac{\cancel{7} \times 3 \times \cancel{x} \times x^2}{\cancel{7} \times 2 \times \cancel{x}} = \frac{3x^2}{2}$$

Simplify $\dfrac{36x}{12x^3}$

Your solution

Answer

Factorising and cancelling common factors gives:

$$\frac{36x}{12x^3} = \frac{12 \times 3 \times x}{12 \times x \times x^2} = \frac{3}{x^2}$$

Example 50

Simplify $\dfrac{3x+6}{6x+12}$.

Solution

First we factorise the numerator and the denominator to see if there are any common factors.

$$\frac{3x+6}{6x+12} = \frac{3(x+2)}{6(x+2)} = \frac{3}{6} = \frac{1}{2}$$

The factors $x+2$ and 3 have been cancelled.

Task

Simplify $\dfrac{12}{2x+8}$.

Your solution

$$\frac{12}{2x+8} =$$

Answer

Factorise the numerator and denominator, and cancel any common factors. $\dfrac{6\times 2}{2(x+4)} = \dfrac{6}{x+4}$

Example 51

Show that the algebraic fraction $\dfrac{3}{x+1}$ and $\dfrac{3(x+4)}{x^2+5x+4}$ are equivalent.

Solution

The denominator, x^2+5x+4, can be factorised as $(x+1)(x+4)$ so that

$$\frac{3(x+4)}{x^2+5x+4} = \frac{3(x+4)}{(x+1)(x+4)}$$

Note that $(x+4)$ is a factor common to both the numerator and the denominator and can be cancelled to leave $\dfrac{3}{x+1}$. Thus $\dfrac{3}{x+1}$ and $\dfrac{3(x+4)}{x^2+5x+4}$ are equivalent fractions.

Task

Show that $\dfrac{x-1}{x^2-3x+2}$ is equivalent to $\dfrac{1}{x-2}$.

First factorise the denominator:

Your solution

$x^2-3x+2=$

Answer

$(x-1)(x-2)$

Now identify the factor common to both numerator and denominator and cancel this common factor:

Your solution

$\dfrac{x-1}{(x-1)(x-2)}=$

Answer

$\dfrac{1}{x-2}$. Hence the two given fractions are equivalent.

Example 52

Simplify $\dfrac{6(4-8x)(x-2)}{1-2x}$

Solution

The factor $4-8x$ can be factorised to $4(1-2x)$. Thus

$$\frac{6(4-8x)(x-2)}{1-2x}=\frac{(6)(4)(1-2x)(x-2)}{(1-2x)}=24(x-2)$$

Task

Simplify $\dfrac{x^2+2x-15}{2x^2-5x-3}$

First factorise the numerator and factorise the denominator:

Your solution

$\dfrac{x^2+2x-15}{2x^2-5x-3}=$

Answer
$\dfrac{(x+5)(x-3)}{(2x+1)(x-3)}$

Then cancel any common factors:

Your solution
$\dfrac{(x+5)(x-3)}{(2x+1)(x-3)}=$

Answer
$\dfrac{x+5}{2x+1}$

Exercises

1. Simplify, if possible,

   (a) $\dfrac{19}{38}$, (b) $\dfrac{14}{28}$, (c) $\dfrac{35}{40}$, (d) $\dfrac{7}{11}$, (e) $\dfrac{14}{56}$

2. Simplify, if possible, (a) $\dfrac{14}{21}$, (b) $\dfrac{36}{96}$, (c) $\dfrac{13}{52}$, (d) $\dfrac{52}{13}$

3. Simplify (a) $\dfrac{5z}{z}$, (b) $\dfrac{25z}{5z}$, (c) $\dfrac{5}{25z^2}$, (d) $\dfrac{5z}{25z^2}$

4. Simplify

   (a) $\dfrac{4x}{3x}$, (b) $\dfrac{15x}{x^2}$, (c) $\dfrac{4s}{s^3}$, (d) $\dfrac{21x^4}{7x^3}$

5. Simplify, if possible,

   (a) $\dfrac{x+1}{2(x+1)}$, (b) $\dfrac{x+1}{2x+2}$, (c) $\dfrac{2(x+1)}{x+1}$, (d) $\dfrac{3x+3}{x+1}$, (e) $\dfrac{5x-15}{5}$, (f) $\dfrac{5x-15}{x-3}$.

6. Simplify, if possible,

   (a) $\dfrac{5x+15}{25x+5}$, (b) $\dfrac{5x+15}{25x}$, (c) $\dfrac{5x+15}{25}$, (d) $\dfrac{5x+15}{25x+1}$

7. Simplify (a) $\dfrac{x^2+10x+9}{x^2+8x-9}$, (b) $\dfrac{x^2-9}{x^2+4x-21}$, (c) $\dfrac{2x^2-x-1}{2x^2+5x+2}$,

   (d) $\dfrac{3x^2-4x+1}{x^2-x}$, (e) $\dfrac{5z^2-20z}{2z-8}$

8. Simplify (a) $\dfrac{6}{3x+9}$, (b) $\dfrac{2x}{4x^2+2x}$, (c) $\dfrac{3x^2}{15x^3+10x^2}$

9. Simplify (a) $\dfrac{x^2-1}{x^2+5x+4}$, (b) $\dfrac{x^2+5x+6}{x^2+x-6}$.

Answers

1. (a) $\frac{1}{2}$, (b) $\frac{1}{2}$, (c) $\frac{7}{8}$, (d) $\frac{7}{11}$, (e) $\frac{1}{4}$.

2. (a) $\frac{2}{3}$, (b) $\frac{3}{8}$, (c) $\frac{1}{4}$, (d) 4

3. (a) 5, (b) 5, (c) $\frac{1}{5z^2}$, (d) $\frac{1}{5z}$.

4. (a) $\frac{4}{3}$, (b) $\frac{15}{x}$, (c) $\frac{4}{s^2}$, (d) $3x$

5. (a) $\frac{1}{2}$, (b) $\frac{1}{2}$, (c) 2, (d) 3, (e) $x-3$, (f) 5

6. (a) $\frac{x+3}{5x+1}$, (b) $\frac{x+3}{5x}$, (c) $\frac{x+3}{5}$, (d) $\frac{5(x+3)}{25x+1}$

7. (a) $\frac{x+1}{x-1}$, (b) $\frac{x+3}{x+7}$, (c) $\frac{x-1}{x+2}$, (d) $\frac{3x-1}{x}$, (e) $\frac{5z}{2}$

8. (a) $\frac{2}{x+3}$, (b) $\frac{1}{2x+1}$, (c) $\frac{3}{5(3x+2)}$.

9. (a) $\frac{x-1}{x+4}$, (b) $\frac{x+2}{x-2}$.

2. Multiplication and division of algebraic fractions

To multiply together two fractions (numerical or algebraic) we multiply their numerators together and then multiply their denominators together. That is

Key Point 19

Multiplication of fractions

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Any factors common to both numerator and denominator can be cancelled. This cancellation can be performed before or after the multiplication.

To divide one fraction by another (numerical or algebraic) we invert the second fraction and then multiply.

Key Point 20

Division of fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \qquad b \neq 0, \quad c \neq 0, \quad d \neq 0$$

Example 53

Simplify (a) $\dfrac{2a}{c} \times \dfrac{4}{c}$, (b) $\dfrac{2a}{c} \times \dfrac{c}{4}$, (c) $\dfrac{2a}{c} \div \dfrac{4}{c}$

Solution

(a) $\dfrac{2a}{c} \times \dfrac{4}{c} = \dfrac{8a}{c^2}$

(b) $\dfrac{2a}{c} \times \dfrac{c}{4} = \dfrac{2ac}{4c} = \dfrac{2a}{4} = \dfrac{a}{2}$

(c) Division is performed by inverting the second fraction and then multiplying.

$\dfrac{2a}{c} \div \dfrac{4}{c} = \dfrac{2a}{c} \times \dfrac{c}{4} = \dfrac{a}{2}$ (from the result in (b))

Example 54

Simplify (a) $\dfrac{1}{5x} \times 3x$, (b) $\dfrac{1}{x} \times x$.

Solution

(a) Note that $3x = \dfrac{3x}{1}$. Then $\dfrac{1}{5x} \times 3x = \dfrac{1}{5x} \times \dfrac{3x}{1} = \dfrac{3x}{5x} = \dfrac{3}{5}$

(b) x can be written as $\dfrac{x}{1}$. Then $\dfrac{1}{x} \times x = \dfrac{1}{x} \times \dfrac{x}{1} = \dfrac{x}{x} = 1$

Simplify (a) $\dfrac{1}{y} \times x$,   (b) $\dfrac{y}{x} \times x$.

Your solution

Answer

(a) $\dfrac{1}{y} \times x = \dfrac{1}{y} \times \dfrac{x}{1} = \dfrac{x}{y}$

(b) $\dfrac{y}{x} \times x = \dfrac{y}{x} \times \dfrac{x}{1} = \dfrac{yx}{x} = y$

Example 55

Simplify $\dfrac{\dfrac{2x}{y}}{\dfrac{3x}{2y}}$

Solution

We can write the fraction as $\dfrac{2x}{y} \div \dfrac{3x}{2y}$.

Inverting the second fraction and multiplying we find

$$\frac{2x}{y} \times \frac{2y}{3x} = \frac{4xy}{3xy} = \frac{4}{3}$$

Example 56
Simplify $\dfrac{4x+2}{x^2+4x+3} \times \dfrac{x+3}{7x+5}$

Solution

Factorising the numerator and denominator we find

$$\frac{4x+2}{x^2+4x+3} \times \frac{x+3}{7x+5} = \frac{2(2x+1)}{(x+1)(x+3)} \times \frac{x+3}{7x+5} = \frac{2(2x+1)(x+3)}{(x+1)(x+3)(7x+5)}$$
$$= \frac{2(2x+1)}{(x+1)(7x+5)}$$

It is usually better to factorise first and cancel any common factors before multiplying. Don't remove any brackets unnecessarily otherwise common factors will be difficult to spot.

Task

Simplify

$$\frac{15}{3x-1} \div \frac{3}{2x+1}$$

Your solution

Answer
To divide we invert the second fraction and multiply:

$$\frac{15}{3x-1} \div \frac{3}{2x+1} = \frac{15}{3x-1} \times \frac{2x+1}{3} = \frac{(5)(3)(2x+1)}{3(3x-1)} = \frac{5(2x+1)}{3x-1}$$

Exercises

1. Simplify (a) $\frac{5}{9} \times \frac{3}{2}$, (b) $\frac{14}{3} \times \frac{3}{9}$, (c) $\frac{6}{11} \times \frac{3}{4}$, (d) $\frac{4}{7} \times \frac{28}{3}$

2. Simplify (a) $\frac{5}{9} \div \frac{3}{2}$, (b) $\frac{14}{3} \div \frac{3}{9}$, (c) $\frac{6}{11} \div \frac{3}{4}$, (d) $\frac{4}{7} \div \frac{28}{3}$

3. Simplify

   (a) $2 \times \frac{x+y}{3}$, (b) $\frac{1}{3} \times 2(x+y)$, (c) $\frac{2}{3} \times (x+y)$

4. Simplify

   (a) $3 \times \frac{x+4}{7}$, (b) $\frac{1}{7} \times 3(x+4)$, (c) $\frac{3}{7} \times (x+4)$, (d) $\frac{x}{y} \times \frac{x+1}{y+1}$, (e) $\frac{1}{y} \times \frac{x^2+x}{y+1}$,

   (f) $\frac{\pi d^2}{4} \times \frac{Q}{\pi d^2}$, (g) $\frac{Q}{\pi d^2/4}$

5. Simplify $\frac{6/7}{s+3}$

6. Simplify $\frac{3}{x+2} \div \frac{x}{2x+4}$

7. Simplify $\frac{5}{2x+1} \div \frac{x}{3x-1}$

Answers

1. (a) $\frac{5}{6}$, (b) $\frac{14}{9}$, (c) $\frac{9}{22}$, (d) $\frac{16}{3}$

2. (a) $\frac{10}{27}$, (b) 14, (c) $\frac{8}{11}$, (d) $\frac{3}{49}$

3. (a) $\frac{2(x+y)}{3}$, (b) $\frac{2(x+y)}{3}$, (c) $\frac{2(x+y)}{3}$

4. (a) $\frac{3(x+4)}{7}$, (b) $\frac{3(x+4)}{7}$, (c) $\frac{3(x+4)}{7}$, (d) $\frac{x(x+1)}{y(y+1)}$, (e) $\frac{x(x+1)}{y(y+1)}$, (f) $Q/4$,
   (g) $\frac{4Q}{\pi d^2}$

5. $\frac{6}{7(s+3)}$

6. $\frac{6}{x}$

7. $\frac{5(3x-1)}{x(2x+1)}$

3. Addition and subtraction of algebraic fractions

To add two algebraic fractions the **lowest common denominator** must be found first. This is the simplest algebraic expression that has the given denominators as its factors. All fractions must be written with this lowest common denominator. Their sum is found by adding the numerators and dividing the result by the lowest common denominator.

To subtract two fractions the process is similar. The fractions are written with the lowest common denominator. The difference is found by subtracting the numerators and dividing the result by the lowest common denominator.

Example 57

State the simplest expression which has $x + 1$ and $x + 4$ as its factors.

Solution

The simplest expression is $(x + 1)(x + 4)$. Note that both $x + 1$ and $x + 4$ are factors.

Example 58

State the simplest expression which has $x - 1$ and $(x - 1)^2$ as its factors.

Solution

The simplest expression is $(x - 1)^2$. Clearly $(x - 1)^2$ must be a factor of this expression. Also, because we can write $(x - 1)^2 = (x - 1)(x - 1)$ it follows that $x - 1$ is a factor too.

Example 59

Express as a single fraction $\dfrac{3}{x+1}+\dfrac{2}{x+4}$

Solution

The simplest expression which has both denominators as its factors is $(x+1)(x+4)$. This is the lowest common denominator. Both fractions must be written using this denominator. Note that $\dfrac{3}{x+1}$ is equivalent to $\dfrac{3(x+4)}{(x+1)(x+4)}$ and $\dfrac{2}{x+4}$ is equivalent to $\dfrac{2(x+1)}{(x+1)(x+4)}$. Thus writing both fractions with the same denominator we have

$$\frac{3}{x+1}+\frac{2}{x+4}=\frac{3(x+4)}{(x+1)(x+4)}+\frac{2(x+1)}{(x+1)(x+4)}$$

The sum is found by adding the numerators and dividing the result by the lowest common denominator.

$$\frac{3(x+4)}{(x+1)(x+4)}+\frac{2(x+1)}{(x+1)(x+4)}=\frac{3(x+4)+2(x+1)}{(x+1)(x+4)}=\frac{5x+14}{(x+1)(x+4)}$$

Key Point 21

Addition of two algebraic fractions

Step 1: Find the lowest common denominator

Step 2: Express each fraction with this denominator

Step 3: Add the numerators and divide the result by the lowest common denominator

Example 60

Express $\dfrac{1}{x-1}+\dfrac{5}{(x-1)^2}$ as a single fraction.

Solution

The simplest expression having both denominators as its factors is $(x-1)^2$. We write both fractions with this denominator.

$$\frac{1}{x-1}+\frac{5}{(x-1)^2}=\frac{x-1}{(x-1)^2}+\frac{5}{(x-1)^2}=\frac{x-1+5}{(x-1)^2}=\frac{x+4}{(x-1)^2}$$

Task

Express $\dfrac{3}{x+7}+\dfrac{5}{x+2}$ as a single fraction.

First find the lowest common denominator:

Your solution

Answer

$(x+7)(x+2)$

Re-write both fractions using this lowest common denominator:

Your solution

$\dfrac{3}{x+7}+\dfrac{5}{x+2}=$

Answer

$$\frac{3(x+2)}{(x+7)(x+2)}+\frac{5(x+7)}{(x+7)(x+2)}$$

Finally, add the numerators and simplify:

Your solution

$\dfrac{3}{x+7}+\dfrac{5}{x+2}=$

Answer

$$\frac{8x+41}{(x+7)(x+2)}$$

Example 61

Express $\dfrac{5x}{7}-\dfrac{3x-4}{2}$ as a single fraction.

Solution

In this example both denominators are simply numbers. The lowest common denominator is 14, and both fractions are re-written with this denominator. Thus

$$\frac{5x}{7}-\frac{3x-4}{2}=\frac{10x}{14}-\frac{7(3x-4)}{14}=\frac{10x-7(3x-4)}{14}=\frac{28-11x}{14}$$

Task

Express $\dfrac{1}{x}+\dfrac{1}{y}$ as a single fraction.

Your solution

Answer

The simplest expression which has x and y as its factors is xy. This is the lowest common denominator. Both fractions are written using this denominator. Noting that $\dfrac{1}{x}=\dfrac{y}{xy}$ and that $\dfrac{1}{y}=\dfrac{x}{xy}$ we find

$$\frac{1}{x}+\frac{1}{y}=\frac{y}{xy}+\frac{x}{xy}=\frac{y+x}{xy}$$

No cancellation is now possible because neither x nor y is a factor of the numerator.

Exercises

1. Simplify (a) $\dfrac{x}{4}+\dfrac{x}{7}$, (b) $\dfrac{2x}{5}+\dfrac{x}{9}$, (c) $\dfrac{2x}{3}-\dfrac{3x}{4}$, (d) $\dfrac{x}{x+1}-\dfrac{2}{x+2}$, (e) $\dfrac{x+1}{x}+\dfrac{3}{x+2}$,

   (f) $\dfrac{2x+1}{3}-\dfrac{x}{2}$, (g) $\dfrac{x+3}{2x+1}-\dfrac{x}{3}$, (h) $\dfrac{x}{4}-\dfrac{x}{5}$

2. Find

   (a) $\dfrac{1}{x+2}+\dfrac{2}{x+3}$, (b) $\dfrac{2}{x+3}+\dfrac{5}{x+1}$, (c) $\dfrac{2}{2x+1}-\dfrac{3}{3x+2}$, (d) $\dfrac{x+1}{x+3}+\dfrac{x+4}{x+2}$,

   (e) $\dfrac{x-1}{x-3}+\dfrac{x-1}{(x-3)^2}$.

3. Find $\dfrac{5}{2x+3}+\dfrac{4}{(2x+3)^2}$.

4. Find $\dfrac{1}{7}s+\dfrac{11}{21}$

5. Express $\dfrac{A}{2x+3}+\dfrac{B}{x+1}$ as a single fraction.

6 Express $\dfrac{A}{2x+5}+\dfrac{B}{(x-1)}+\dfrac{C}{(x-1)^2}$ as a single fraction.

7 Express $\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$ as a single fraction.

8 Express $\dfrac{Ax+B}{x^2+x+10}+\dfrac{C}{x-1}$ as a single fraction.

9 Express $Ax+B+\dfrac{C}{x+1}$ as a single fraction.

10 Show that $\dfrac{x_1}{\dfrac{1}{x_3}-\dfrac{1}{x_2}}$ is equal to $\dfrac{x_1x_2x_3}{x_2-x_3}$.

11 Find (a) $\dfrac{3x}{4}-\dfrac{x}{5}+\dfrac{x}{3}$, (b) $\dfrac{3x}{4}-\left(\dfrac{x}{5}+\dfrac{x}{3}\right)$.

Answers

1. (a) $\dfrac{11x}{28}$, (b) $\dfrac{23x}{45}$, (c) $-\dfrac{x}{12}$, (d) $\dfrac{x^2-2}{(x+1)(x+2)}$, (e) $\dfrac{x^2+6x+2}{x(x+2)}$,

   (f) $\dfrac{x+2}{6}$, (g) $\dfrac{9+2x-2x^2}{3(2x+1)}$, (h) $\dfrac{x}{20}$

2. (a) $\dfrac{3x+7}{(x+2)(x+3)}$, (b) $\dfrac{7x+17}{(x+3)(x+1)}$, (c) $\dfrac{1}{(2x+1)(3x+2)}$,

   (d) $\dfrac{2x^2+10x+14}{(x+3)(x+2)}$, (e) $\dfrac{x^2-3x+2}{(x-3)^2}$

3. $\dfrac{10x+19}{(2x+3)^2}$

4. $\dfrac{3s+11}{21}$

5. $\dfrac{A(x+1)+B(2x+3)}{(2x+3)(x+1)}$

6. $\dfrac{A(x-1)^2+B(x-1)(2x+5)+C(2x+5)}{(2x+5)(x-1)^2}$

7. $\dfrac{A(x+1)+B}{(x+1)^2}$

8. $\dfrac{(Ax+B)(x-1)+C(x^2+x+10)}{(x-1)(x^2+x+10)}$

9. $\dfrac{(Ax+B)(x+1)+C}{x+1}$

11. (a) $\dfrac{53x}{60}$, (b) $\dfrac{13x}{60}$

Formulae and Transposition

1.5

Introduction

Formulae are used frequently in almost all aspects of engineering in order to relate a physical quantity to one or more others. Many well-known physical laws are described using formulae. For example, you may have already seen Ohm's law, $v = iR$, or Newton's second law of motion, $F = ma$.

In this Section we describe the process of evaluating a formula, explain what is meant by the **subject** of a formula, and show how a formula is rearranged or transposed. These are basic skills required in all aspects of engineering.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide algebraic fractions

Learning Outcomes

On completion you should be able to ...

- evaluate a formula
- transpose a formula

1. Using formulae and substitution

In the study of engineering, physical quantities can be related to each other using a formula. The formula will contain variables and constants which represent the physical quantities. To evaluate a formula we must **substitute** numbers in place of the variables.

For example, Ohm's law provides a formula for relating the voltage, v, across a resistor with resistance value, R, to the current through it, i. The formula states

$$v = iR$$

We can use this formula to calculate v if we know values for i and R. For example, if $i = 13$ A, and $R = 5\,\Omega$, then

$$\begin{aligned} v &= iR \\ &= (13)(5) \\ &= 65 \end{aligned}$$

The voltage is 65 V.

Note that it is important to pay attention to the units of any physical quantities involved. Unless a consistent set of units is used a formula is not valid.

Example 62

The kinetic energy, K, of an object of mass M moving with speed v can be calculated from the formula, $K = \frac{1}{2}Mv^2$.

Calculate the kinetic energy of an object of mass 5 kg moving with a speed of 2 m s^{-1}.

Solution

In this example $M = 5$ and $v = 2$. Substituting these values into the formula we find

$$\begin{aligned} K &= \frac{1}{2}Mv^2 \\ &= \frac{1}{2}(5)(2^2) \\ &= 10 \end{aligned}$$

In the SI system the unit of energy is the joule. Hence the kinetic energy of the object is 10 joules.

Task

The area, A, of the circle of radius r can be calculated from the formula $A = \pi r^2$.

If we know the diameter of the circle, d, we can use the equivalent formula $A = \dfrac{\pi d^2}{4}$. Find the area of a circle having diameter 0.1 m. Your calculator will be preprogrammed with the value of π.

Your solution

$A =$

Answer

$\dfrac{\pi(0.1)^2}{4} = 0.0079$ m^2

Example 63

The volume, V, of a circular cylinder is equal to its cross-sectional area, A, times its length, h.

Find the volume of a cylinder having diameter 0.1 m and length 0.3 m.

Solution

We can use the result of the previous Task to obtain the cross-sectional area $A = \dfrac{\pi d^2}{4}$. Then

$$\begin{aligned} V &= Ah \\ &= \frac{\pi(0.1)^2}{4} \times 0.3 \\ &= 0.0024 \end{aligned}$$

The volume is 0.0024 m^3.

2. Rearranging a formula

In the formula for the area of a circle, $A = \pi r^2$, we say that A is the **subject** of the formula. A variable is the subject of the formula if it appears by itself on one side of the formula, usually the left-hand side, and **nowhere else in the formula**. If we are asked to **transpose** the formula for r, or **solve** for r, then we have to make r the subject of the formula. When transposing a formula *whatever is done to one side is done to the other*. There are five rules that must be adhered to.

Key Point 22

Rearranging a formula

You may carry out the following operations

- add the same quantity to both sides of the formula
- subtract the same quantity from both sides of the formula
- multiply both sides of the formula by the same quantity
- divide both sides of the formula by the same quantity
- take a 'function' of both sides of the formula: for example, find the reciprocal of both sides (i.e. invert).

Example 64

Transpose the formula $p = 5t - 17$ for t.

Solution

We must obtain t on its own on the left-hand side. We do this in stages by using one or more of the five rules in Key Point 22. For example, by adding 17 to both sides of $p = 5t - 17$ we find

$$p + 17 = 5t - 17 + 17$$

so that $$p + 17 = 5t$$

Dividing both sides by 5 we obtain t on its own:

$$\frac{p+17}{5} = t$$

so that $t = \dfrac{p+17}{5}$.

Example 65

Transpose the formula $\sqrt{2q} = p$ for q.

Solution

First we square both sides to remove the square root. Note that $(\sqrt{2q})^2 = 2q$. This gives

$$2q = p^2$$

Second we divide both sides by 2 to get $q = \frac{p^2}{2}$.

Note that in general by squaring both sides of an equation may introduce extra solutions not valid for the original equation. In Example 65 if $p = 2$ then $q = 2$ is the only solution. However, if we transform to $q = \dfrac{p^2}{2}$, then if $q = 2$, p can be $+2$ or -2.

Task

Transpose the formula $v = \sqrt{t^2 + w}$ for w.

You must obtain w on its own on the left-hand side. Do this in several stages.

First square both sides to remove the square root:

Your solution

Answer

$v^2 = t^2 + w$

Then, subtract t^2 from both sides to obtain an expression for w:

Your solution

Answer

$v^2 - t^2 = w$

Finally, write down the formula for w:

Your solution

Answer

$w = v^2 - t^2$

Example 66

Transpose $x = \dfrac{1}{y}$ for y.

Solution

We must try to obtain an expression for y. Multiplying both sides by y has the effect of removing this fraction:

$$\begin{array}{rrcll} \text{Multiply both sides of} & x & = & \dfrac{1}{y} & \text{by } y \text{ to get} \\ & yx & = & y \times \dfrac{1}{y} & \\ \text{so that} & yx & = & 1 & \end{array}$$

Divide both sides by x to leaves y on its own, $y = \dfrac{1}{x}$.

Alternatively: simply invert both sides of the equation $x = \dfrac{1}{y}$ to get $\dfrac{1}{x} = y$.

Example 67

Make R the subject of the formula

$$\frac{2}{R} = \frac{3}{x+y}$$

Solution

In the given form R appears in a fraction. Inverting both sides gives

$$\frac{R}{2} = \frac{x+y}{3}$$

Thus multiplying both sides by 2 gives

$$R = \frac{2(x+y)}{3}$$

Task

Make R the subject of the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

(a) Add the two terms on the right:

Your solution

Answer

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2}$$

(b) Write down the complete formula:

Your solution

Answer

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

(c) Now invert both sides:

Your solution

Answer

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

Engineering Example 2

Heat flow in an insulated metal plate

Introduction

Thermal insulation is important in many domestic (e.g. central heating) and industrial (e.g cooling and heating) situations. Although many real situations involve heat flow in more than one dimension, we consider only a one dimensional case here. The flow of heat is determined by temperature and thermal conductivity. It is possible to model the amount of heat Q (J) crossing point x in one dimension (the heat flow in the x direction) from temperature T_2 (K) to temperature T_1 (K) (in which $T_2 > T_1$) in time t s by

$$\frac{Q}{t} = \lambda A \left(\frac{T_2 - T_1}{x} \right),$$

where λ is the thermal conductivity in W m^{-1} K.

Problem in words

Suppose that the upper and lower sides of a metal plate connecting two containers are insulated and one end is maintained at a temperature T_2 (K) (see Figure 7).

The plate is assumed to be infinite in the direction perpendicular to the sheet of paper.

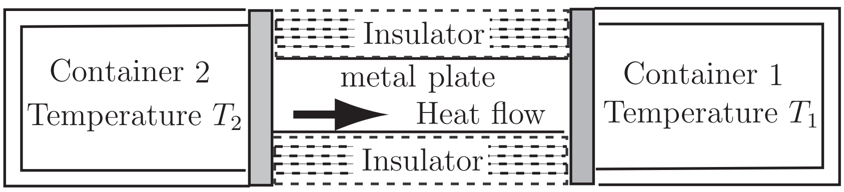


Figure 7: A laterally insulated metal plate

(a) Find a formula for T.

(b) If $\lambda = 205$ (W m^{-1} K^{-1}), $T_1 = 300$ (K), $A = 0.004$ (m^2), $x = 0.5$ (m), calculate the value of T_2 required to achieve a heat flow of 100 J s^{-1}.

Mathematical statement of the problem

(a) Given $\dfrac{Q}{t} = \lambda A \left(\dfrac{T_2 - T_1}{x} \right)$ express T_2 as the subject of the formula.

(b) In the formula found in part (a) substitute $\lambda = 205,\ T_1 = 300,\ A = 0.004$, $x = 0.5$ and $\dfrac{Q}{t} = 100$ to find T_2.

Mathematical analysis

(a) $\dfrac{Q}{t} = \lambda A\left(\dfrac{T_2 - T_1}{x}\right)$

Divide both sides by λA

$$\frac{Q}{t\lambda A} = \frac{T_2 - T_1}{x}$$

Multiply both sides by x

$$\frac{Qx}{t\lambda A} = T_2 - T_1$$

Add T_1 to both sides

$$\frac{Qx}{t\lambda A} + T_1 = T_2$$

which is equivalent to

$$T_2 = \frac{Qx}{t\lambda A} + T_1$$

(b) Substitute $\lambda = 205,\ T_1 = 300,\ A = 0.004,\ x = 0.5$ and $\dfrac{Q}{t} = 100$ to find T_2:

$$T_2 = \frac{100 \times 0.5}{205 \times 0.004} + 300 \approx 60.9 + 300 = 360.9$$

So the temperature in container 2 is 361 K to 3 sig.fig.

Interpretation

The formula $T_2 = \dfrac{Qx}{t\lambda A} + T_1$ can be used to find a value for T_2 that would achieve any desired heat flow. In the example given T_2 would need to be about 361 K ($\approx 78°$C) to produce a heat flow of 100 J s^{-1}.

Exercises

1. The formula for the volume of a cylinder is $V = \pi r^2 h$. Find V when $r = 5$ cm and $h = 15$ cm.

2. If $R = 5p^2$, find R when (a) $p = 10$, (b) $p = 16$.

3. For the following formulae, find y at the given values of x.

   (a) $y = 3x + 2$, $\quad x = -1$, $x = 0$, $x = 1$.

   (b) $y = -4x + 7$, $\quad x = -2$, $x = 0$, $x = 1$.

   (c) $y = x^2$, $\quad x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$.

4. If $P = \dfrac{3}{QR}$ find P if $Q = 15$ and $R = 0.300$.

5. If $y = \sqrt{\dfrac{x}{z}}$ find y if $x = 13.200$ and $z = 15.600$.

6. Evaluate $M = \dfrac{\pi}{2r + s}$ when $r = 23.700$ and $s = -0.2$.

7. To convert a length measured in metres to one measured in centimetres, the length in metres is multiplied by 100. Convert the following lengths to cm. (a) 5 m, (b) 0.5 m, (c) 56.2 m.

8. To convert an area measured in m^2 to one measured in cm^2, the area in m^2 is multiplied by 10^4. Convert the following areas to cm^2. (a) 5 m^2, (b) 0.33 m^2, (c) 6.2 m^2.

9. To convert a volume measured in m^3 to one measured in cm^3, the volume in m^3 is multiplied by 10^6. Convert the following volumes to cm^3. (a) 15 m^3, (b) 0.25 m^3, (c) 8.2 m^3.

10. If $\eta = \dfrac{4Q_P}{\pi d^2 L n}$ evaluate η when $Q_P = 0.0003$, $d = 0.05$, $L = 0.1$ and $n = 2$.

11. The moment of inertia of an object is a measure of its resistance to rotation. It depends upon both the mass of the object and the distribution of mass about the axis of rotation. It can be shown that the moment of inertia, J, of a solid disc rotating about an axis through its centre and perpendicular to the plane of the disc, is given by the formula

$$J = \frac{1}{2} M a^2$$

    where M is the mass of the disc and a is its radius. Find the moment of inertia of a disc of mass 12 kg and diameter 10 m. The SI unit of moment of inertia is $\text{kg}\,\text{m}^2$.

12. Transpose the given formulae to make the given variable the subject.

    (a) $y = 3x - 7$, for x, (b) $8y + 3x = 4$, for x, (c) $8x + 3y = 4$ for y,

    (d) $13 - 2x - 7y = 0$ for x.

13. Transpose the formula $PV = RT$ for (a) V, (b) P, (c) R, (d) T.

14. Transpose $v = \sqrt{x + 2y}$, (a) for x, (b) for y.

15. Transpose $8u + 4v - 3w = 17$ for each of u, v and w.

16. When a ball is dropped from rest onto a horizontal surface it will bounce before eventually coming to rest after a time T where

$$T = \frac{2v}{g}\left(\frac{1}{1-e}\right)$$

where v is the speed immediately after the first impact, and g is a constant called the acceleration due to gravity. Transpose this formula to make e, the coefficient of restitution, the subject.

17. Transpose $q = A_1\sqrt{\dfrac{2gh}{(A_1/A_2)^2 - 1}}$ for A_2.

18. Make x the subject of (a) $y = \dfrac{r+x}{1-rx}$, (b) $y = \sqrt{\dfrac{x-1}{x+1}}$.

19. In the design of orifice plate flowmeters, the volumetric flowrate, Q ($\text{m}^3\ \text{s}^{-1}$), is given by

$$Q = C_d A_o \sqrt{\frac{2g\Delta h}{1 - A_o^2/A_p^2}}$$

where C_d is a dimensionless discharge coefficient, Δh (m) is the head difference across the orifice plate and A_o (m^2) is the area of the orifice and A_p (m^2) is the area of the pipe.

(a) Rearrange the equation to solve for the area of the orifice, A_o, in terms of the other variables.

(b) A volumetric flowrate of 100 $\text{cm}^3\ \text{s}^{-1}$ passes through a 10 cm inside diameter pipe. Assuming a discharge coefficient of 0.6, calculate the required orifice diameter, so that the head difference across the orifice plate is 200 mm.

[Hint: be very careful with the units!]

Answers

1. 1178.1 cm^3
2. (a) 500, (b) 1280
3. (a) $-1, 2, 5$, (b) $15, 7, 3$, (c) 5,3,1,0,
4. P=0.667
5. $y = 0.920$
6. $M = 0.067$
7. (a) 500 cm, (b) 50 cm, (c) 5620 cm.
8. (a) 50000 cm^2, (b) 3300 cm^2, (c) 62000 cm^2.
9. (a) 15000000 cm^3, (b) 250000 cm^3, (c) 8200000 cm^3.
10. $\eta = 0.764$.
11. 150 kg m^2
12. (a) $x = \dfrac{y+7}{3}$, (b) $x = \dfrac{4-8y}{3}$, (c) $y = \dfrac{4-8x}{3}$, (d) $x = \dfrac{13-7y}{2}$
13. (a) $V = \dfrac{RT}{P}$, (b) $P = \dfrac{RT}{V}$, (c) $R = \dfrac{PV}{T}$, (d) $T = \dfrac{PV}{R}$
14. (a) $x = v^2 - 2y$, (b) $y = \dfrac{v^2 - x}{2}$
15. $u = \dfrac{17 - 4v + 3w}{8}$, $v = \dfrac{17 - 8u + 3w}{4}$, $w = \dfrac{8u + 4v - 17}{3}$
16. $e = 1 - \dfrac{2v}{gT}$
17. $A_2 = \pm\sqrt{\dfrac{A_1^2 q^2}{2A_1^2 gh + q^2}}$
18. (a) $x = \dfrac{y - r}{1 + yr}$, (b) $x = \dfrac{1 + y^2}{1 - y^2}$
19.

(a) $A_0 = \dfrac{QA_p}{\sqrt{Q^2 + 2g\Delta h A_p^2 C_d^2}}$

(b) $Q = 100$ cm^3 s^{-1}$= 10^{-4}$ m^2 s^{-1}

$A_p = \pi\dfrac{0.1^2}{4} = 0.007854$ m^2

$C_d = 0.6$

$\Delta h = 0.2$ m

$g = 9.81$ m s^{-2}

Substituting in answer (a) gives

$A_o = 8.4132 \times 10^{-5}$ m^2

so diameter $= \sqrt{\dfrac{4A_o}{\pi}} = 0.01035$ m $= 1.035$ cm

NOTES

Index for Workbook 1

Workbook 1

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